Superdiffusions and positive solutions of nonlinear partial differential equations

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Contents

Preface

This book is devoted to the applications of the probability theory to the theory of nonlinear partial differential equations. More precisely, we investigate the class $\mathcal{U}$ of all positive solutions of the equation $Lu = \psi(u) \quad \text{in } E$ where L is an elliptic differential operator of the second order, E is a bounded smooth domain in $\mathbb{R}^d$ and ψ is a continuously differentiable positive function.

The progress in solving this problem till the beginning of 2002 was described in the monograph [**D**]. [We use an abbreviation [**D**] for [Dy02].] Under mild conditions on ψ, a trace on the boundary ∂E was associated with every $u \in \mathcal{U}$. This is a pair (Γ, ν) where Γ is a subset of ∂E and ν is a σ-finite measure on $\partial E \setminus \Gamma$. [A point y belongs to Γ if $\psi'(u)$ tends sufficiently fast to infinity as $x \to y$.] All possible values of the trace were described and a 1-1 correspondence was established between these values and a class of solutions called σ-moderate. We say that u is σ-moderate if it is the limit of an increasing sequence of moderate solutions. [A moderate solution is a solution u such that $u \le h$ where $Lh = 0$ in E.] In the Epilogue to [**D**], a crucial outstanding question was formulated: *Are all the solutions σ-moderate?* In the case of the equation $\Delta u = u^2$ in a domain of class C^4, a positive answer to this question was given in the thesis of Mselati [Ms02a] - a student of J.-F. Le Gall. [1] However his principal tool - the Brownian snake - is not applicable to more general equations. In a series of publications by Dynkin and Kuznetsov [Dy04b], [Dy04c], [Dy04d], [Dy04e],[DK03], [DK04], [Ku04], Mselati's result was extended, by using a superdiffusion instead of the snake, to the equation $\Delta u = u^\alpha \quad \text{with } 1 < \alpha \le 2$. This required an enhancement of the superdiffusion theory which can be of interest for anybody who works on application of probabilistic methods to mathematical analysis.

The goal of this book is to give a self-contained presentation of these new developments. The book may be considered as a continuation of the monograph [**D**]. In the first three chapters we give an overview of the theory presented in [**D**] without duplicating the proofs which can be found in [**D**]. The book can be read independently of [**D**]. [It might be even useful to read the first three chapters before reading [**D**].]

In a series of papers (including [MV98a], [MV98b] and [MV04]) M. Marcus and L. Véron investigated positive solutions of the equation $\Delta u = u^\alpha$

[1] The dissertation of Mselati was published in 2004 (see [Ms04]).

by purely analytic methods. Both, analytic and probabilistic approach have their advantages and an interaction between analysts and probabilists was important for the progress of the field. I take this opportunity to thank M. Marcus and L. Véron for keeping me informed about their work.

The Choquet capacities are one of the principal tools in the study of the equation $\Delta u = u^\alpha$. This class contains the Poisson capacities used in the work of Dynkin and Kuznetsov and in this book and the Bessel capacities used by Marcus and Véron and by other analysts. I am very grateful to I. E. Verbitsky who agreed to write Appendix B, where the relations between the Poisson and Bessel capacities are established, thus allowing to connect the work of both groups.

I am indebted to S. E. Kuznetsov who provided me several preliminary drafts of his paper [Ku04] used in Chapters **8** and **9**. I am grateful to him and to J.-F. Le Gall and B. Mselati for many helpful discussions. It is my pleasant duty to thank J.-F. Le Gall for a permission to include into the book as the Appendix his note which clarifies a statement used but not proved in Mselati's thesis (we use it in Chapter **8**).

I am especially indebted to Yuan-chung Sheu for reading carefully the entire manuscript and suggesting many corrections and improvements.

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CHAPTER 1

Introduction

1. Trace theory

1.1. We consider a differential equation

$$Lu = \psi(u) \quad \text{in } E \tag{1.1}$$

where E is a domain in $\mathbb{R}^d$, L is a uniformly elliptic differential operator in E and ψ is a function from $[0,\infty)$ to $[0,\infty)$. Under various conditions on E, L and ψ [1] we investigate the set $\mathcal{U}$ of all positive solutions of (1.1). Our base is the trace theory presented in [**D**]. Here we give a brief description of this theory (which is applicable to an arbitrary domain E and a wide class of functions ψ described in Section 4.3). [2]

1.2. Moderate and σ-moderate solutions. Our starting point is the representation of positive solutions of the linear equation

$$Lh = 0 \quad \text{in } E \tag{1.2}$$

by Poisson integrals. If E is smooth [3] and if $k(x,y)$ is the *Poisson kernel* [4] of L in E, then the formula

$$h_\nu(x) = \int_{\partial E} k(x,y)\nu(dy) \tag{1.3}$$

establishes a 1-1 correspondence between the set $\mathcal{M}(\partial E)$ of all finite measures ν on ∂E and the set $\mathcal{H}$ of all positive solutions of (1.2). (We call solutions of (1.2) *harmonic functions.*)

A solution u is called *moderate* if it is dominated by a harmonic function. There exists a 1-1 correspondence between the set $\mathcal{U}_1$ of all moderate solutions and a subset $\mathcal{H}_1$ of $\mathcal{H}$: $h \in \mathcal{H}_1$ is the minimal harmonic function dominating $u \in \mathcal{U}_1$, and u is the maximal solution dominated by h. We put $\nu \in \mathcal{N}_1$ if $h_\nu \in \mathcal{H}_1$. We denote by u_ν the element of $\mathcal{U}_1$ corresponding to h_ν.

An element u of $\mathcal{U}$ is called *σ-moderate solutions* if there exist $u_n \in \mathcal{U}_1$ such that $u_n(x) \uparrow u(x)$ for all x. The labeling of moderate solutions by

[1] We discuss these condtions in Section 4.

[2] It is applicable also to functions $\psi(x,u)$ depending on $x \in E$.

[3] We use the name smooth for open sets of class $C^{2,\lambda}$ unless another class is indicated explicitely.

[4] For an arbitrary domain, $k(x,y)$ should be replaced by the Martin kernel and ∂E should be replaced by a certain Borel subset E' of the Martin boundary (see Chapter 7 in [**D**]).

measures $\nu \in \mathcal{N}_1$ can be extended to σ-moderate solutions by the convention: if $\nu_n \in \mathcal{N}_1$, $\nu_n \uparrow \nu$ and if $u_{\nu_n} \uparrow u$, then put $\nu \in \mathcal{N}_0$ and $u = u_\nu$.

1.3. Lattice structure in $\mathcal{U}$. [5] We write $u \le v$ if $u(x) \le v(x)$ for all $x \in E$. This determines a partial order in $\mathcal{U}$. For every $\tilde{\mathcal{U}} \subset \mathcal{U}$, there exists a unique element u of $\mathcal{U}$ with the properties: (a) $u \ge v$ for every $v \in \tilde{\mathcal{U}}$; (b) if $\tilde{u} \in \mathcal{U}$ satisfies (a), then $u \le \tilde{u}$. We denote this element $\operatorname{Sup} \tilde{\mathcal{U}}$.

For every $u, v \in \mathcal{U}$, we put $u \oplus v = \operatorname{Sup} W$ where W is the set of all $w \in \mathcal{U}$ such that $w \le u + v$. Note that $u \oplus v$ is moderate if u and v are moderate and it is σ-moderate if so are u and v.

In general, $\operatorname{Sup} \tilde{\mathcal{U}}$ does not coincide with the pointwise supremum (the latter does not belong to $\mathcal{U}$). However, both are equal if $\tilde{\mathcal{U}}$ is closed under $\oplus$. Moreover, in this case there exist $u_n \in \tilde{\mathcal{U}}$ such that $u_n(x) \uparrow u(x)$ for all $x \in E$. Therefore, if $\tilde{\mathcal{U}}$ is closed under $\oplus$ and it consists of moderate solutions, then $\operatorname{Sup} \tilde{\mathcal{U}}$ is σ-moderate. In particular, to every Borel subset Γ of ∂E there corresponds a σ-moderate solution

$$u_\Gamma = \operatorname{Sup}\{u_\nu : \nu \in \mathcal{N}_1, \nu \text{ is concentrated on } \Gamma\}. \tag{1.4}$$

We also associate with Γ another solution w_Γ. First, we define w_K for closed K by the formula

$$w_K = \operatorname{Sup}\{u \in \mathcal{U} : u = 0 \quad \text{on } \partial E \setminus K\}. \tag{1.5}$$

For every Borel subset Γ of ∂E, we put

$$w_\Gamma = \operatorname{Sup}\{w_K : \text{closed } K \subset \Gamma\}. \tag{1.6}$$

Proving that $u_\Gamma = w_\Gamma$ was a key part of the program outlined in [**D**].

1.4. Singular points of a solution u. We consider classical solutions of (1.1) which are twice continuously differentiable in E. However they can tend to infinity as $x \to y \in \partial E$. We say that y is a singular point of u if it is a point of rapid growth of $\psi'(u)$. [A special role of $\psi'(u)$ is due to the fact that the tangent space to $\mathcal{U}$ at point u is described by the equation $Lv = \psi'(u)v$.]

The rapid growth of a positive continuous function $a(x)$ can be defined analytically or probabilistically. The analytic definition involves the Poisson kernel (or Martin kernel) $k_a(x, y)$ of the operator $Lu - au$: $y \in \partial E$ is a point of rapid growth for a if $k_a(x, y) = 0$ for all $x \in E$. A more transparent probabilistic definition is given in Chapter 3.

We say that a Borel subset Γ of ∂E is f-closed if Γ contains all singular points of the solution u_Γ defined by (1.4).

[5] See Chapter 8, Section 5 in [**D**].

1.5. Definition and properties of trace. The trace of $u \in \mathcal{U}$ (which we denote $\mathrm{Tr}(u)$) is defined as a pair (Γ, ν) where Γ is the set of all singular points of u and ν is a measure on $\partial E \setminus \Gamma$ given by the formula

$$\nu(B) = \sup\{\mu(B) : \mu \in \mathcal{N}_1, \mu(\Gamma) = 0, u_\mu \leq u\}. \tag{1.7}$$

We have

$$u_\nu = \mathrm{Sup}\{\text{ moderate } u_\mu \leq u \text{ with } \mu(\Gamma) = 0\}$$

and therefore u_ν is σ-moderate.

The trace of every solution u has the following properties:

1.5.A. Γ is a Borel f-closed set; ν is a σ-finite measure of class $\mathcal{N}_0$ such that $\nu(\Gamma) = 0$ and all singular points of u_ν belong to Γ.

1.5.B. If $\mathrm{Tr}(u) = (\Gamma, \nu)$, then

$$u \geq u_\Gamma \oplus u_\nu. \tag{1.8}$$

Moreover, $u_\Gamma \oplus u_\nu$ is the maximal σ-moderate solution dominated by u.

1.5.C. If (Γ, ν) satisfies the condition 1.5.A, then $\mathrm{Tr}(u_\Gamma \oplus u_\nu) = (\Gamma', \nu)$, the symmetric difference between Γ and Γ' is not charged by any measure $\mu \in \mathcal{N}_1$. Moreover, $u_\Gamma \oplus u_\nu$ is the minimal solution with this property and the only one which is σ-moderate.

2. Organizing the book

Let $u \in \mathcal{U}$ and let $\mathrm{Tr}(u) = (\Gamma, \nu)$. The proof that u is σ-moderate consists of three parts:

A. $u \geq u_\Gamma \oplus u_\nu$.

B. $u_\Gamma = w_\Gamma$.

C. $u \leq w_\Gamma \oplus u_\nu$.

It follows from A–C that $u = u_\Gamma \oplus u_\nu$ and therefore u is σ-moderate because u_Γ and u_ν are σ-moderate.

We already have obtained A as a part of the trace theory (see (1.8)) which covers a general equation (1.1). Parts B and C will be covered for the equation $\Delta = u^\alpha$ with $1 < \alpha \leq 2$. To this end we use, beside the trace theory, a number of analytic and probabilistic tools. In Chapters **2** and **3** we survey a part of these tools (mostly related to the theory of superdiffusion) already prepared in [**D**]. A recent enhancement of the superdifusion theory –the $\mathbb{N}$-measures – is presented in Chapter **4**. Another new tool – bounds for the Poisson capacities – is the subject of Chapter **6**. By using all these tools, we prove in Chapter **7** a basic inequality for superdiffusions which makes it possible to prove (in Chapter **8**) that $u_\Gamma = w_\Gamma$ (Part B) and therefore w_Γ is σ-moderate. The concluding part C is proved in Chapter **9** by using absolute continuity results on superdiffusions presented in Chapter **5**. In Chapter **8** we use an upper estimate of w_K in terms of the Poisson capacity established by S. E. Kuznetsov [Ku04]. In the Appendix contributed by J.-F. Le Gall a property of the Brownian motion is proved which is also used in Chapter **8**.

Notes at the end of each chapter describe the relation of its contents to the literature on the subject.

3. Notation

3.1. We use notation $C^k(D)$ for the set of k times continously differentiable function on D and we write $C(D)$ for $C^0(D)$. We put $f \in C^\lambda(D)$ if there exists a constant Λ such that $|f(x) - f(y)| \leq \Lambda|x-y|^\lambda$ for all $x, y \in D$ (Hölder continuity). Notation $C^{k,\lambda}(D)$ is used for the class of k times differentiable functions with all partials of order k belonging to $C^\lambda(D)$.

We write $f \in \mathcal{B}$ if f is a positive $\mathcal{B}$-measurable function. Writing $f \in b\mathcal{B}$ means that, in addition, f is bounded.

For every subset D of $\mathbb{R}^d$ we denote by $\mathcal{B}(D)$ the Borel σ-algebra in D.

We write $D \Subset E$ if $\bar{D}$ is a compact subset of E. We say that a sequence D_n *exhausts* E if $D_1 \Subset D_2 \Subset \cdots \Subset D_n \Subset \ldots$ and E is the union of D_n.

$\mathcal{D}_i$ stands for the partial derivative $\frac{\partial}{\partial x_i}$ with respect to the coordinate x_i of x and $\mathcal{D}_{ij}$ means $\mathcal{D}_i\mathcal{D}_j$.

We denote by $\mathcal{M}(E)$ the set of all finite measures on E and by $\mathcal{P}(E)$ the set of all probability measures on E. We write $\langle f, \mu\rangle$ for the integral of f with respect to μ.

$\delta_y(B) = 1_B(y)$ is the unit mass concentrated at y.

A kernel from a measurable space $(E_1, \mathcal{B}_1)$ to a measurable space $(E_2, \mathcal{B}_2)$ is a function $K(x, B)$ such that $K(x, \cdot)$ is a finite measure on $\mathcal{B}_2$ for every $x \in E_1$ and $K(\cdot, B)$ is an $\mathcal{B}_1$-measurable function for every $B \in \mathcal{B}_2$.

If u is a function on an open set E and if $y \in \partial E$, then writing $u(y) = a$ means $u(x) \to a$ as $x \to y, x \in E$.

We put

$$\operatorname{diam}(B) = \sup\{|x-y| : x, y \in B\} \quad \text{(the diameter of } B\text{)},$$
$$d(x, B) = \inf_{y \in B} |x - y| \quad \text{(the distance from x to B)},$$
$$\rho(x) = d(x, \partial E) \quad \text{for } x \in E.$$

We denote by C constants depending only on E, L and ψ (their values can vary even within one line). We indicate explicitely the dependence on any additional parameter. For instance, we write C_κ for a constant depending on a parameter κ (besides a possible dependence on E, L, ψ).

4. Assumptions

4.1. Operator L. There are several levels of assumptions used in this book.

In the most general setting, we consider a second order differential operator

$$Lu(x) = \sum_{i,j=1}^{d} a_{ij}(x)\mathcal{D}_{ij}u(x) + \sum_{i=1}^{d} b_i(x)\mathcal{D}_i u(x) \tag{4.1}$$

in a domain E in $\mathbb{R}^d$. Without loss of generality we can put $a_{ij} = a_{ji}$. We assume that

4.1.A. [Uniform ellipticity] There exists a constant $\kappa > 0$ such that

$$\sum a_{ij}(x)t_i t_j \geq \kappa \sum t_i^2 \quad \text{for all } x \in E, t_1, \dots, t_d \in \mathbb{R}.$$

4.1.B. All coefficients $a_{ij}(x)$ and $b_i(x)$ are bounded and Hölder continuous.

In a part of the book we assume that L is of divergence form

$$Lu(x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} u(x). \tag{4.2}$$

In Chapters **8** and **9** we restrict ourselves to the Laplacian $\Delta = \sum_1^d \mathcal{D}_i^2$.

4.2. Domain E. Mostly we assume that E is a bounded *smooth domain.* This name is used for domains of class $C^{2,\lambda}$ which means that ∂E can be straightened near every point $x \in \partial E$ by a diffeomorphism ϕ_x of class $C^{2,\lambda}$. To define straightening, we consider a half-space $E_+ = \{x = (x_1, \dots, x_d) : x_d > 0\} = \mathbb{R}^{d-1} \times (0, \infty)$. Denote E_0 its boundary $\{x = (x_1, \dots, x_d) : x_d = 0\}$. We assume that, for every $x \in \partial E$, there exists a ball $B(x, \varepsilon) = \{y : |x-y| < \varepsilon\}$ and a diffeomorphism ϕ_x from $B(x, \varepsilon)$ onto a domain $\tilde{E} \subset \mathbb{R}^d$ such that $\phi_x(B(x,\varepsilon) \cap E) \subset E_+$ and $\phi_x(B(x,\varepsilon) \cap \partial E) \subset E_0$. (We say that ϕ_x straightens the boundary in $B(x,\varepsilon)$.) The Jacobian of ϕ_x does not vanish and we can assume that it is strictly positive.

Main results of Chapters **8** and **9** depend on an upper bound for w_K established in [Ku04] for domains of class C^4. All results of Chapters **8** and **9** can be automatically extended to domains of class $C^{2,\lambda}$ if the bound for w_K will be proved for such domains.

4.3. Function ψ. In general we assume that ψ is a function on $[0, \infty)$ with the properties:

4.3.A. $\psi \in C^2(\mathbb{R}_+)$.

4.3.B. $\psi(0) = \psi'(0) = 0$, $\psi''(u) > 0$ for $u > 0$.

[It follows from 4.3.B that ψ is monotone and convex and ψ' is bounded on each interval $[0, t]$.]

4.3.C. There is a constant a such that

$$\psi(2u) \leq a\psi(u)$$

for all u.

4.3.D. $\int_N^\infty ds \ \left[\int_0^s \psi(u)\, du\right]^{-1/2} < \infty$ for some $N > 0$.

Keller [Ke57] and Osserman [Os57] proved independently that this condition implies that functions $u \in \mathcal{U}(E)$ are uniformly bounded on every set $D \Subset E$. [6]

In Chapters **7-9** we assume that

$$\psi(u) = u^\alpha, \ 1 < \alpha \le 2. \tag{4.3}$$

(In Chapter **6** we do not need the restriction $\alpha \le 2$.)

5. Notes

The trace $\mathrm{Tr}(u)$ was introduced in [Ku98] and [DK98b] under the name the fine trace. We suggested to use the name "rough trace" for a version of the trace considered before in the literature. (In the monograph **[D]** the rough trace is treated in Chapter 10 and the fine trace is introduced and studied in Chapter 11.)

The most publications were devoted to the equation

$$\Delta u = u^\alpha, \alpha > 1. \tag{5.1}$$

In the subcritical case $1 < d < \frac{\alpha+1}{\alpha-1}$, the rough trace coincides with the fine trace and it determines a solution of (5.1) uniquely. As it was shown by Le Gall, this is not true in the supercritical case: $d \ge \frac{\alpha+1}{\alpha-1}$.

In a pioneering paper [GV91] Gmira and Véron proved that, in the subcritical case, the generalized Dirichlet problem

$$\begin{aligned} \Delta u &= u^\alpha \quad \text{in } E,\\ u &= \mu \quad \text{on } \partial E \end{aligned} \tag{5.2}$$

has a unique solution for every finite measure μ. (In our notation, this is u_μ.)

A program of investigating $\mathcal{U}$ by using a superdiffusion was initiated in [Dy91a]. In [Dy94] Dynkin conjectured that, for every $1 < \alpha \le 2$ and every d, the problem (5.2) has a solution if and only if μ does not charge sets which are, a.s., not hit by the range of the superdiffusion. [7] [The conjecture was proved, first, in the case $\alpha = 2$, by Le Gall and then, for all $1 < \alpha \le 2$, by Dynkin and Kuznetsov .]

A classification of all positive solutions of $\Delta u = u^2$ in the unit disk $E = \{x \in \mathbb{R}^2 : |x| < 1\}$ was announced by Le Gall in [Le93]. [This is also a subcritical case.] The result was proved and extended to a wide class of smooth planar domains in [Le97]. Instead of a superdiffusion Le Gall used his own invention – a path-valued process called the Brownian snake. He established a 1-1 correspondence between $\mathcal{U}$ and pairs (Γ, ν) where Γ is a closed subset of ∂E and ν is a Radon measure on $\partial E \setminus \Gamma$.

Dynkin and Kuznetsov [DK98a] extended Le Gall's results to the equation $Lu = u^\alpha, 1 < \alpha \le 2$. They introduced a rough boundary trace for

[6] In a more general setting this is proved in **[D]**, Section **5**.3.

[7] The restriction $\alpha \le 2$ is needed because a related superdiffusion exists only in this range.

solutions of this equation. They described all possible values of the trace and they represented the maximal solution with a given trace in terms of a superdiffusion.

Marcus and Véron [MV98a]–[MV98b] investigated the rough traces of solutions by purely analytic means. They extended the theory to the case $\alpha > 2$ and they proved that the rough trace determines a solution uniquely in the subcritical case.

The theory of fine trace developed in [DK98b] provided a classification of all σ-moderate soltions. Mselati's dissertation [Ms02a] finalized the classification for the equation $\Delta u = u^2$ by demonstrating that, in this case, all solutions are σ-moderate. A substantial enhancement of the superdiffusion theory was necessary to get similar results for a more general equation $\Delta u = u^\alpha$ with $1 < \alpha \le 2$.

CHAPTER 2

Analytic approach

In this chapter we consider equation **1**.(1.1) under minimal assumptions on L, ψ and E: conditions **1**.4.1.A– **1**.4.1.B for L, conditions **1**.4.3.A–**1**.4.3.D for ψ and and assumption that E is bounded and belongs to class $C^{2,\lambda}$.

For every open subset D of E we define an operator V_D that maps positive Borel functions on ∂D to positive solutions of the equation $Lu = \psi(u)$ in D. If D is smooth and f is continuous, then $V_D(f)$ is a solution of the boundary value problem

$$\begin{aligned} Lu = \psi(u) \quad & \text{in } D, \\ u = f \quad & \text{on } \partial D. \end{aligned}$$

In general, $u = V_D(f)$ is a solution of the integral equation

$$u + G_D\psi(u) = K_D f$$

where G_D and K_D are the Green and Poisson operators for L in D. Operators V_D have the properties:

$$\begin{aligned} V_D(f) \le V_D(\tilde{f}) \quad & \text{if } f \le \tilde{f}, \\ V_D(f_n) \uparrow V_D(f) \quad & \text{if } f_n \uparrow f, \\ V_D(f_1 + f_2) \le V_D(f_1) + V_D(f_2). & \end{aligned}$$

The Comparison principle plays for the equation **1**.(1.1) a role similar to the role of the Maximum principle for linear elliptic equations. There is also an analog of the Mean value property: if $u \in \mathcal{U}(E)$, then $V_D(u) = u$ for every $D \Subset E$. The set $\mathcal{U}(E)$ of all positive solutions is closed under Sup and under pointwise convergence.

We label moderate solutions by measures ν on ∂E belonging to a class $\mathcal{N}_1^E$ and we label σ-moderate solutions by a wider class $\mathcal{N}_0^E$. A special role is played by $\nu \in \mathcal{N}_0^E$ taking only values 0 and ∞.

An algebraic approach to the equation **1**.(1.1) is discussed in Section 3. In Section 4 we introduce the Choquet capacities which play a crucial role in subsequent chapters.

Most propositions stated in Chapters 2 and 3 are proved in [**D**]. In each case we give an exact reference to the corresponding place in [**D**]. We provide a complete proof for every statement not proved in [**D**].

1. Operators G_D and K_D

1.1. Green function and Green operator. Suppose that D is a bounded smooth domain and that L satisfies conditions **1**.4.1.A–**1**.4.1.B. Then there exists a unique continuous function g_D from $\bar{D} \times \bar{D}$ to $[0, \infty]$ such that, for every $f \in C^\lambda(D)$,

$$u(x) = \int_D g_D(x,y) f(y) dy \tag{1.1}$$

is the unique solution of the problem

$$\begin{aligned} Lu &= -f \quad \text{in } D, \\ u &= 0 \quad \text{on } \partial D. \end{aligned} \tag{1.2}$$

The function g_D is called the *Green function*. It has the following properties:

1.1.A. For every $y \in D$, $u(x) = g_D(x,y)$ is a solution of the problem

$$\begin{aligned} Lu &= 0 \quad \text{in } D \setminus \{y\}, \\ u &= 0 \quad \text{on } \partial D. \end{aligned} \tag{1.3}$$

1.1.B. For all $x, y \in D$,

$$0 < g_D(x,y) \le C\Gamma(x-y) \tag{1.4}$$

where C is a constant depending only on D and L and [1]

$$\Gamma(x) = \begin{cases} |x|^{2-d} & \text{for } d \ge 3, \\ (-\log|x|) \vee 1 & \text{for } d = 2, \\ 1 & \text{for } d = 1. \end{cases} \tag{1.5}$$

If L is of divergence form and $d \ge 3$, then

$$g_D(x,y) \le C\rho(x)|x-y|^{1-d}, \tag{1.6}$$

$$g_D(x,y) \le C\rho(x)\rho(y)|x-y|^{-d}. \tag{1.7}$$

[See [GrW82].]

The *Green operator* is defined by the formula (1.1).

1.2. Poisson kernel and Poisson operator. Suppose that D is a bounded smooth domain and let γ be the *normalized surface area* on ∂D. The *Poisson kernel* k_D is a continuous function from $D \times \partial D$ to $(0, \infty)$ with the property: for every $\varphi \in C(D)$,

$$h(x) = \int_{\partial D} k_D(x,y)\varphi(y)\gamma(dy) \tag{1.8}$$

is a unique solution of the problem

$$\begin{aligned} Lu &= 0 \quad \text{in } D, \\ u &= \varphi \quad \text{on } \partial D. \end{aligned} \tag{1.9}$$

[1] There is a misprint in the expression for $\Gamma(x)$ in [**D**], page 88.

We have the following bounds for the Poisson kernel: [2]

$$C^{-1}\rho(x)|x-y|^{-d} \le k_D(x,y) \le C\rho(x)|x-y|^{-d} \tag{1.10}$$

where

$$\rho(x) = \operatorname{dist}(x, \partial D). \tag{1.11}$$

The *Poisson operator* K_D is defined by the formula (1.8).

2. Operator V_D and equation $Lu = \psi(u)$

2.1. Operator V_D. By Theorem 4.3.1 in [**D**], if ψ satisfies conditions **1**.4.3.B and **1**.4.3.C, then, for every $f \in b\mathcal{B}(\bar{E})$ and for every open subset D of E, there exists a unique solution of the equation

$$u + G_D\psi(u) = K_D f. \tag{2.1}$$

We denote it $V_D(f)$. It follows from (2.1) that:

2.1.A. $V_D(f) \le K_D(f)$, in particular, $V_D(c) \le c$ for every constant c.

We have:

2.1.B. [[**D**], 4.3.2.A] If $f \le \tilde{f}$, then $V_D(f) \le V_D(\tilde{f})$.

2.1.C. [[**D**], 4.3.2.C] If $f_n \uparrow f$, then $V_D(f_n) \uparrow V_D(f)$.

Properties 2.1.B and 2.1.C allow to define $V_D(f)$ for all $f \in \mathcal{B}(\bar{D})$ by the formula

$$V_D(f) = \sup_n V_D(f \wedge n). \tag{2.2}$$

The extended operators satisfy equation (2.1) and conditions 2.1.A-2.1.C. They have the properties:

2.1.D. [[**D**], Theorem 8.2.1] For every $f_1, f_2 \in \mathcal{B}(D)$,

$$V_D(f_1 + f_2) \le V_D(f_1) + V_D(f_2). \tag{2.3}$$

2.1.E. [[**D**], 8.2.1.J] For every D and every $f \in \mathcal{B}(\partial D)$, the function $u = V_D(f)$ is a solution of the equation

$$Lu = \psi(u) \quad \text{in } D. \tag{2.4}$$

We denote by $\mathcal{U}(D)$ the set of all positive solutions of the equation (2.4).

[2]See, e.g. [Ma75], Lemma 6 and the Appendix B in [**D**].

2.2. Properties of $\mathcal{U}(D)$. We have:

2.2.A. [[**D**], 8.2.1.J and 8.2.1.H] If D is smooth and if f is continuous in a neighborhood O of $\tilde{x} \in \partial D$, then $V_D f(x) \to f(\tilde{x})$ at $x \to \tilde{x}, x \in D$. If D is smooth and bounded and if a function $f : \partial D \to [0, \infty)$ is continuous, then $u = V_D(f)$ is a unique solution of the problem

$$\begin{aligned} Lu &= \psi(u) \quad \text{in } D, \\ u &= f \quad \text{on } \partial D. \end{aligned} \tag{2.5}$$

2.2.B. (Comparison principle)[[**D**], 8.2.1.H.] Suppose D is bounded. Then $u \leq v$ assuming that $u, v \in C^2(D)$,

$$Lu - \psi(u) \geq Lv - \psi(v) \quad \text{in } D \tag{2.6}$$

and, for every $\tilde{x} \in \partial D$,

$$\limsup [u(x) - v(x)] \leq 0 \quad \text{as } x \to \tilde{x}. \tag{2.7}$$

2.2.C. (Mean value property)[[**D**], 8.2.1.D] If $u \in \mathcal{U}(D)$, then, for every $U \Subset D$, $V_U(u) = u$ in D (which is equivalent to the condition $u + G_U \psi(u) = K_U u$).

2.2.D. [[**D**], Theorem 5.3.2] If $u_n \in \mathcal{U}(E)$ converge pointwise to u, then u belongs to $\mathcal{U}(E)$.

2.2.E. [[**D**], Theorem 5.3.1] For every pair $D \Subset E$ there exists a constant b such that $u(x) \leq b$ for all $u \in \mathcal{U}(E)$ and all $x \in D$. [3]

The next two propositions are immediate implications of the Comparison principle.

We say that $u \in C^2(E)$ is a *supersolution* if $Lu \leq \psi(u)$ in E and that it is a *subsolution* if $Lu \geq \psi(u)$ in E. Every $h \in \mathcal{H}(E)$ is a supersolution because $Lh = 0 \leq \psi(h)$. It follows from 2.2.B that:

2.2.F. If a subsolution u and a supersolution v satisfy (2.7), then $u \leq v$ in E.

2.2.G. If $\psi(u) = u^\alpha$ with $\alpha > 1$, then, for every $u \in \mathcal{U}(D)$ and for all $x \in D$,

$$u(x) \leq C d(x, \partial D)^{-2/(\alpha-1)}.$$

Indeed, if $d(x, \partial D) = \rho$, then the ball $B = \{y : |y - x| < \rho\}$ is contained in D. Function $v(y) = C(\rho^2 - |y - x|^2)^{-2/(\alpha-1)}$ is equal to ∞ on ∂B and, for sufficiently large C, $Lv(y) - v(y)^\alpha \leq 0$ in B. [4] By 2.2.B, $u \leq v$ in B. In particular, $u(x) \leq v(x) = C\rho^{-2/(\alpha-1)}$.

[3] As we already have mentioned, this is an implication of **1**.4.3.D.

[4] See, e.g., [Dy91a], page 102, or [**D**], page 71.

2.3. On moderate solutions. Recall that an element u of $\mathcal{U}(E)$ is called *moderate* if $u \le h$ for some $h \in \mathcal{H}(E)$. The formula

$$u + G_E\psi(u) = h \tag{2.8}$$

establishes a 1-1 correspondence between the set $\mathcal{U}_1(E)$ of moderate elements of $\mathcal{U}(E)$ and a subset $\mathcal{H}_1(E)$ of $\mathcal{H}(E)$: h is the minimal harmonic function dominating u, and u is the maximal solution dominated by h. Formula **1**.(1.3) defines a 1-1 correspondence $\nu \leftrightarrow h_\nu$ between $\mathcal{M}(\partial E)$ and $\mathcal{H}(E)$. We put $\nu \in \mathcal{N}_1^E$ if $h_\nu \in \mathcal{H}_1(E)$ and we denote u_ν the moderate solution corresponding to $\nu \in \mathcal{N}_1^E$. In this notation,

$$u_\nu + G_E\psi(u_\nu) = h_\nu. \tag{2.9}$$

(The correspondence $\nu \leftrightarrow u_\nu$ is 1-1 and monotonic.)

We need the following properties of $\mathcal{N}_1^E$, $\mathcal{H}_1(E)$ and $\mathcal{U}_1(E)$.

2.3.A. [Corollary 3.1 in [**D**], Section 8.3.2] If $h \in \mathcal{H}_1(E)$ and if $h' \le h$ belongs to $\mathcal{H}(E)$, then $h' \in \mathcal{H}_1(E)$. Therefore $\mathcal{N}_1^E$ contains with ν all measures $\nu' \le \nu$.

2.3.B. [[**D**],Theorem 8.3.3] $\mathcal{H}_1(E)$ is a convex cone (that is it is closed under addition and under multiplication by positive numbers).

2.3.C. If Γ is a closed subset of ∂E and if $\nu \in \mathcal{M}(E)$ is concentrated on Γ, then $h_\nu = 0$ on $\partial E \setminus \Gamma$.

Indeed, it follows from **1**.(1.3) and (1.10) that

$$h_\nu(x) \le C\rho(x)\int_\Gamma |x-y|^{-d}\nu(dy).$$

2.3.D. If $\nu \in \mathcal{N}_1^E$ and Γ is a closed subset of ∂E, then $u_\nu = 0$ on $O = \partial E\setminus\Gamma$ if and only if $\nu(O) = 0$.

PROOF. If $\nu(O) = 0$, then $h_\nu = 0$ on O by 2.3.C, and $u_\nu = 0$ on O because $u_\nu \le h_\nu$ by (2.8).

On the other hand, if $u_\nu = 0$ on O, then $\nu(K) = 0$ for every closed subset K of O. Indeed, if η is the restriction of ν to K, then $u_\eta = 0$ on Γ because $\Gamma \subset \partial E \setminus K$ and $\eta(\partial E \setminus K) = 0$. We also have $u_\eta \le u_\nu = 0$ on O. Hence $u_\eta = 0$ on ∂E. The Comparison principle 2.2.B implies that $u_\eta = 0$. Therefore $\eta = 0$. □

2.3.E. [[**D**], Proposition 12.2.1.A] [5] If $h \in \mathcal{H}(E)$ and if $G_E\psi(h)(x) < \infty$ for some $x \in E$, then $h \in \mathcal{H}_1(E)$.

[5] Proposition 12.2.1.A is stated for $\psi(u) = u^\alpha$ but the proof is applicable to a general ψ.

2.3.F. (Extended mean value property) If $U \subset D$ and if $\nu \in \mathcal{N}_1^D$ is concentrated on Γ such that $\bar{\Gamma} \cap \bar{U} = \emptyset$, then $V_U(u_\nu) = u_\nu$.

If $u \in \mathcal{U}_1(D)$ vanishes on $\partial D \setminus \Gamma$, then $V_U(u) = u$ for every $U \subset D$ such that $\bar{\Gamma} \cap \bar{U} = \emptyset$.

The first part is Theorem 8.4.1 in [**D**]. The second part follows from the first one because $u \in \mathcal{U}_1(D)$ is equal to u_ν for some $\nu \in \mathcal{N}_1^D$ and, by 2.3.D, $\nu(\partial D \setminus \bar{\Gamma}) = 0$.

2.3.G. Suppose that $\nu \in \mathcal{N}_1^E$ is supported by a closed set $K \subset \partial E$ and let $E_\varepsilon = \{x \in E : d(x, K) > \varepsilon\}$. Then

$$u^\varepsilon = V_{E_\varepsilon}(h_\nu) \downarrow u_\nu \quad \text{as } \varepsilon \downarrow 0.$$

Proof. Put $V^\varepsilon = V_{E_\varepsilon}$. By (2.9), $h_\nu = u^\varepsilon + G_{E^\varepsilon}\psi(u^\varepsilon) \geq u^\varepsilon$ for every ε. Let $\varepsilon' < \varepsilon$. By applying the second part of 2.3.F to $U = E_\varepsilon, D = E_{\varepsilon'}, u = u^{\varepsilon'}$ and $\Gamma = \partial E_{\varepsilon'} \cap E$ we get $V^\varepsilon(u^{\varepsilon'}) = u^{\varepsilon'}$. By 2.1.B,

$$u^\varepsilon = V^\varepsilon(h_\nu) \geq V^\varepsilon(u^{\varepsilon'}) = u^{\varepsilon'}.$$

Hence u^ε tends to a limit u as $\varepsilon \downarrow 0$. By 2.2.D, $u \in \mathcal{U}(E)$. For every ε, $u^\varepsilon \leq h_\nu$ and therefore $u \leq h_\nu$. On the other hand, if $v \in \mathcal{U}(E)$ and $v \leq h_\nu$, then, by 2.3.F, $v = V^\varepsilon(v) \leq V^\varepsilon(h_\nu) = u^\varepsilon$ and therefore $v \leq u$. Hence, u is a maximal element of $\mathcal{U}(E)$ dominated by h_ν which means that $u = u_\nu$. □

2.4. On σ-moderate solutions. Denote by $\mathcal{U}_0(E)$ the set of all σ-moderate solutions. (Recall that u is σ-moderate if there exist moderate u_n such that $u_n \uparrow u$.) If $\nu_1 \leq \cdots \leq \nu_n \leq \ldots$ is an increasing sequence of measures, then $\nu = \lim \nu_n$ is also a measure. We put $\nu \in \mathcal{N}_0^E$ if $\nu_n \in \mathcal{N}_1^E$. If $\nu \in \mathcal{N}_1^E$, then $\infty \cdot \nu = \lim_{t \uparrow \infty} t\nu$ belongs to $\mathcal{N}_0^E$. Measures $\mu = \infty \cdot \nu$ take only values 0 and ∞ and therefore $c\mu = \mu$ for every $0 < c \leq \infty$. [We put $0 \cdot \infty = 0$.]

Lemma 2.1. *[[**D**], Lemma 8.5.1] There exists a monotone mapping $\nu \to u_\nu$ from $\mathcal{N}_0^E$ onto $\mathcal{U}_0(E)$ such that*

$$u_{\nu_n} \uparrow u_\nu \quad \text{if } \nu_n \uparrow \nu \tag{2.10}$$

and, for $\nu \in \mathcal{N}_1^E$, u_ν is the maximal solution dominated by h_ν

The following properties of $\mathcal{N}_0^E$ are proved on pages 120-121 of [**D**]:

2.4.A. A measure $\nu \in \mathcal{N}_0^E$ belongs to $\mathcal{N}_1^E$ if and only if $\nu(E) < \infty$. If $\nu_n \in \mathcal{N}_1(E)$ and $\nu_n \uparrow \nu \in \mathcal{M}(\partial E)$, then $\nu \in \mathcal{N}_1^E$. [6]

2.4.B. If $\nu \in \mathcal{N}_0^E$ and if $\mu \leq \nu$, then $\mu \in \mathcal{N}_0^E$.

2.4.C. Suppose E is a bounded smooth domain and O is a relatively open subset of ∂E. If $\nu \in \mathcal{N}_0^E$ and $\nu(O) = 0$, then $u_\nu = 0$ on O.

An important class of σ-moderate solutions are u_Γ defined by **1**.(1.4).

[6] See [**D**]. 8.5.4.A.

2.4.D. [[**D**], 8.5.5.A] For every Borel $\Gamma \subset \partial E$, there exists $\nu \in \mathcal{N}_1^E$ concentrated on Γ such that $u_\Gamma = u_{\infty\cdot\nu}$.

2.5. On solution w_Γ. We list some properties of these solutions (defined in the Introduction by (1.5) and (1.6)).

2.5.A. [[**D**], Theorem 5.5.3] If K is a closed subset of ∂E, then w_K defined by **1**.(1.5) vanishes on $\partial E \setminus K$. [It is the maximal element of $\mathcal{U}(E)$ with this property.]

2.5.B. If $\nu \in \mathcal{N}_0^E$ is concentrated on a Borel set Γ, then $u_\nu \leq w_\Gamma$.

Proof. If $\nu \in \mathcal{N}_1^E$ is supported by a compact set K, then $u_\nu = 0$ on $\partial E \setminus K$ by 2.4.C and $u_\nu \leq w_K$ by **1**.(1.5). If $\nu \in \mathcal{N}_0^E$, then there exist $\nu_n \in \mathcal{N}_1^E$ such that $\nu_n \uparrow \nu$. The measures ν_n are also concentrated on Γ and therefore there exists a sequence of compact sets $K_{mn} \subset \Gamma$ such that $\nu_{mn} \uparrow \nu_n$ where ν_{mn} is the restriction of ν_n to K_{mn}. We have $u_{\nu_{mn}} \leq w_{K_{mn}} \leq w_\Gamma$. Hence, $u_\nu \leq w_\Gamma$. □

3. Algebraic approach to the equation $Lu = \psi(u)$

In the Introduction we defined, for every subset $\tilde{\mathcal{U}}$ of $\mathcal{U}(E)$, an element Sup $\tilde{\mathcal{U}}$ of $\mathcal{U}(E)$ and we introduced in $\mathcal{U}(E)$ a semi-group operation $u \oplus v$. In a similar way, we define now Inf $\tilde{\mathcal{U}}$ as the maximal element u of $\mathcal{U}(E)$ such that $u \leq v$ for all $v \in \tilde{U}$. We put, for $u, v \in \mathcal{U}$ such that $u \geq v$,

$$u \ominus v = \operatorname{Inf}\{w \in \mathcal{U} : w \geq u - v\}.$$

Both operations $\oplus$ and $\ominus$ can be expressed through an operator π.

Denote by $C_+(E)$ the class of all positive functions $f \in C(E)$. Put $u \in \mathcal{D}(\pi)$ and $\pi(u) = v$ if $u \in C_+(E)$ and $V_{D_n}(u) \to v$ pointwise for every sequence D_n exhausting E. By 2.1.E and 2.2.D, $\pi(u) \in \mathcal{U}(E)$. It follows from 2.1.B that $\pi(u_1) \leq \pi(u_2)$ if $u_1 \leq u_2$.

Put

$$\mathcal{U}^-(E) = \{u \in C_+(E) : V_D(u) \leq u \quad \text{for all } D \Subset E\}$$

and

$$\mathcal{U}^+(E) = \{u \in C_+(E) : V_D(u) \geq u \quad \text{for all } D \Subset E\}.$$

By 2.2.C, $\mathcal{U}(E) \subset \mathcal{U}^-(E) \cap \mathcal{U}^+(E)$. It follows from the Comparison principle 2.2.B that $\mathcal{U}^-$ contains all supersolutions and $\mathcal{U}^+$ contains all subsolutions. In particular, $\mathcal{H}(E) \subset \mathcal{U}^-(E)$.

For every sequence D_n exhausting E, we have: [see [**D**], 8.5.1.A–8.5.1.D]

3.A. If $u \in \mathcal{U}^-(E)$, then $V_{D_n}(u) \downarrow \pi(u)$ and

$$\pi(u) = \sup\{\tilde{u} \in \mathcal{U}(E) : \tilde{u} \leq u\} \leq u.$$

3.B. If $u \in \mathcal{U}^+(E)$, then $V_{D_n}(u) \uparrow \pi(u)$ and

$$\pi(u) = \inf\{\tilde{u} \in \mathcal{U}(E) : \tilde{u} \geq u\} \geq u.$$

Clearly,

3.C. If $u, v \in \mathcal{U}^+(E)$, then $\max\{u, v\} \in \mathcal{U}^+(E)$. If $u, v \in \mathcal{U}^-(E)$, then $\min\{u, v\} \in \mathcal{U}^-(E)$.

It follows from 2.1.D (subadditivity of V_D) that:

3.D. If $u, v \in \mathcal{U}^-(E)$, then $u + v \in \mathcal{U}^-(E)$. If $u, v \in \mathcal{U}(E)$ and $u \geq v$, then $u - v \in \mathcal{U}^+(E)$.

It is easy to see that:

3.E. If $u, v \in \mathcal{U}(E)$, then $u \oplus v = \pi(u + v)$

3.F. If $u \geq v \in \mathcal{U}(E)$, then $u \ominus v = \pi(u - v)$.

Denote $\mathcal{U}^*(E)$ the minimal convex cone that contains $\mathcal{U}^-(E)$ and $\mathcal{U}^+(E)$.

4. Choquet capacities

Suppose that E is a separable locally compact metrizable space. Denote by $\mathcal{K}$ the class of all compact sets and by $\mathcal{O}$ the class of all open sets in E. A $[0, +\infty]$-valued function Cap on the collection of all subsets of E is called a *capacity* if:

4.A. $\mathrm{Cap}(A) \leq \mathrm{Cap}(B)$ if $A \subset B$.

4.B. $\mathrm{Cap}(A_n) \uparrow \mathrm{Cap}(A)$ if $A_n \uparrow A$.

4.C. $\mathrm{Cap}(K_n) \downarrow \mathrm{Cap}(K)$ if $K_n \downarrow K$ and $K_n \in \mathcal{K}$.

A set B is called capacitable if These conditions imply

(4.1)

$$\mathrm{Cap}(B) = \sup\{\mathrm{Cap}(K) : K \subset B, K \in \mathcal{K}\} = \inf\{\mathrm{Cap}(O) : O \supset B, O \in \mathcal{O}\}.$$

The following results are due to Choquet [Ch54].

I. Every Borel set B is capacitable. [7]

II. Suppose that a function $\mathrm{Cap} : \mathcal{K} \to [0, +\infty]$ satisfies 4.A–4.C and the following condition:

4.D. For every $K_1, K_2 \in \mathcal{K}$,

$$\mathrm{Cap}(K_1 \cup K_2) + \mathrm{Cap}(K_1 \cap K_2) \leq \mathrm{Cap}(K_1) + \mathrm{Cap}(K_2).$$

Then Cap can be extended to a capacity on E.

[7] The relation (4.1) is true for a larger class of analytic sets but we do not use this fact.

5. Notes

The class of moderate solutions was introduced and studied in [DK96a]. σ-moderate solutions, the lattice structure in the space of solutions and the operation $u \oplus v$ appeared, first, in [DK98b] in connection with the fine trace theory. The operation $u \ominus v$ was defined and used by Mselati in [Ms02a].

We defined a moderate solution u_ν as the solution of the integral equation (2.9). If E is a bounded smooth domain, $L = \Delta$ and ψ is a continuous increasing function with $\psi(0) = 0$, then this is equivalent to the conditions: $\int_E [u(x) + d(x, \partial E)\psi(u(x))]dx < \infty$ and, for every $f \in C^{2,\lambda}(\bar{E})$ vanishing at 0,

$$-\int_E [u + \Delta f + \psi(u)f]dx = \int_{pE} \frac{\partial f}{\partial n} d\nu.$$

[Here $\frac{\partial}{\partial n}$ means the derivative in the direction of the outward normal to ∂E.) Analysts call such functions weak solutions of the equation $-\Delta u + \psi(u) = \nu$.

CHAPTER 3

Probabilistic approach

Our base is the theory of diffusions and superdiffusions.

A diffusion describes a random motion of a particle. An example is the Brownian motion in $\mathbb{R}^d$. This is a Markov process with continuous paths and with the transition density

$$p_t(x,y) = (2\pi t)^{-d/2} e^{-|x-y|^2/2t}$$

which is the fundamental solution of the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u.$$

A Brownian motion in a domain E can be obtained by killing the path at the first exit time from E. By replacing $\frac{1}{2}\Delta$ by an operator L of the form **1**.(4.1), we define a Markov process called L-diffusion. We also use an L-diffusion with killing rate ℓ corresponding to the equation

$$\frac{\partial u}{\partial t} = Lu - \ell u$$

and an L-diffusion conditioned to exit from E at a point $y \in \partial E$. The latter can be constructed by the so-called h-transform with $h(x) = k_E(x,y)$.

An (L,ψ)-superdiffusion is a model of random evolution of a cloud of particles. Each particle performs an L-diffusion. It dies at random time leaving a random offspring of the size regulated by the function ψ. All children move independently of each other (and of the family history) with the same transition and procreation mechanism as the parent. Our subject is the family of the exit measures (X_D, P_μ) from open sets $D \subset E$. An idea of this construction is explained on Figure 1 (borrowed from [**D**]).

Here we have a scheme of a process started by two particles located at points x_1, x_2 in D. The first particle produces at its death time two children that survive until they reach ∂D at points y_1, y_2. The second particle has three children. One reaches the boundary at point y_3, the second one dies childless and the third one has two children. Only one of them hits ∂D at point y_4. The initial and exit measure are described by the formulae

$$\mu = \sum \delta_{x_i}, \quad X_D = \sum \delta_{y_i}.$$

To get an (L,ψ)-superdiffusion, we pass to the limit as the mass of each particle and its expected life time tend to 0 and an initial number of particles tends to infinity. We refer for detail to [**D**].

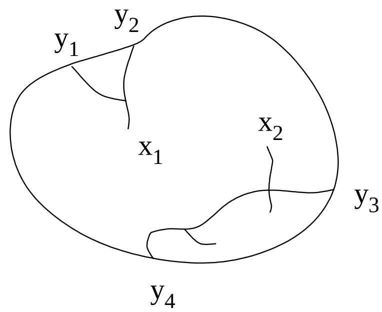

FIGURE 1

We consider superdiffusions as a special case of branching exit Markov systems. Such a system is defined as a family of of exit measures (X_D, P_μ) subject to four conditions, the central two are a Markov property and a continuous branching property. To every right continuous strong Markov process ξ in a metric space E there correspond branching exit Markov systems called superprocesses. Superdiffusions are superprocesses corresponding to diffusions. Superprocesses corresponding to Brownian motions are called super-Brownian motions.

A substantial part of Chapter **3** is devoted to two concepts playing a key role in applications of superdiffusions to partial differential equations: the range of a superprocess and the stochastic boundary values for superdiffusions.

1. Diffusion

1.1. Definition and properties. To every operator L subject to the conditions **1**.4.1.A–**1**.4.1.B there corresponds a strong Markov process $\xi = (\xi_t, \Pi_x)$ in E called an L-*diffusion*. The path ξ_t is defined on a random interval $[0, \tau_E)$. It is continuous and its limit ξ_{τ_E} as $t \to \tau_E$ belongs to ∂E. For every open set $D \subset E$ we denote by τ_D the first exit time of ξ from D.

PROPOSITION 1.1 ([**D**], Lemma 6.2.1). *The function $\Pi_x \tau_D$ is bounded for every bounded domain D.*

There exists a function $p_t(x, y) > 0, t > 0, x, y \in E$ (called the *transition density*) such that:

$$\int_E p_s(x, z)dz\; p_t(z, y) = p_{s+t}(x, y) \quad \text{for all } s, t > 0, x, y \in E$$

and, for every $f \in \mathcal{B}(E)$,

$$\Pi_x f(\xi_t) = \int_E p_t(x, y) f(y)\, dy.$$

An L-diffusion has the following properties:

1.1.A. [[**D**], Sections 6.2.4-6.2.5] If $D \subset E$, then, for every $f \in \mathcal{B}(\bar{E})$,

$$K_D f(x) = \Pi_x f(\xi_{\tau_D}) 1_{\tau_D < \infty}, \quad G_D f(x) = \Pi_x \int_0^{\tau_D} f(\xi_s)\, ds. \tag{1.1}$$

1.1.B. [[**D**], 6.3.2.A.] Suppose that $a \geq 0$ belongs to $C^\lambda(\bar{E})$. If $v \geq 0$ is a soluton of the equation

$$Lv = av \quad \text{in } E, \tag{1.2}$$

then

$$v(x) = \Pi_x v(\xi_{\tau_E}) \exp\left[-\int_0^{\tau_E} a(\xi_s) ds\right]. \tag{1.3}$$

1.1.C. [[**D**], 6.2.5.D.] If $D \subset E$ are two smooth open sets, then

$$k_D(x,y) = k_E(x,y) - \Pi_x 1_{\tau_D < \tau_E} k_E(\xi_{\tau_D}, y) \quad \text{for all } x \in D, y \in \partial E \cap \partial D. \tag{1.4}$$

1.2. Diffusion with killing rate ℓ. An L-diffusion with killing rate ℓ corresponds to a differential operator $Lu - \ell u$. Here ℓ is a positive Borel function. Its the Green and the Poisson operators in a domain D are given by the formulae

$$\begin{aligned} G_D^\ell f(x) &= \Pi_x \int_0^{\tau_D} \exp\left\{-\int_0^t \ell(\xi_s)\, ds\right\} f(\xi_t) dt, \\ K_D^\ell f(x) &= \Pi_x \exp\left\{-\int_0^{\tau_D} \ell(\xi_s)\, ds\right\} f(\xi_{\tau_D}) 1_{\tau_D < \infty}. \end{aligned} \tag{1.5}$$

THEOREM 1.1. *Suppose ξ is an L-diffusion, $\tau = \tau_D$ is the first exit time from a bounded smooth domain D, $\ell \geq 0$ is bounded and belongs to $C^\lambda(D)$. If $\varphi \geq 0$ is a continuous function on ∂D, then $z = K_D^\ell \varphi$ is a unique solution of the integral equation*

$$u + G_D(\ell u) = K_D \varphi. \tag{1.6}$$

If ρ is a bounded Borel function on D, then $\varphi = G_D^\ell \rho$ is a unique solution of the integral equation

$$u + G_D(\ell u) = G_D \rho. \tag{1.7}$$

The first part is proved in [*D*], Theorem 6.3.1. Let us prove the second one. Put $Y_s^t = \exp\{-\int_s^t \ell(\xi_r) dr\}$. Since $\frac{\partial Y_s^t}{\partial s} = \ell(\xi_s) Y_s^t$, we have

$$Y_0^t = 1 - \int_0^t \ell(\xi_s) Y_s^t ds. \tag{1.8}$$

Note that

$$G_D(\ell\varphi)(x) = \Pi_x \int_0^\tau ds \ell(\xi_s) \Pi_{\xi_s} \int_0^\tau Y_0^r \rho(\xi_r) dr.$$

By the Markov property of ξ, the right side is equal to

$$\Pi_x \int_0^\tau ds \ell(\xi_s) \int_s^\tau Y_s^t \rho(\xi_t) dt.$$

By Fubini's theorem and (1.8), this integral is equal to

$$\Pi_x \int_0^\tau dt \rho(\xi_t) \int_0^t \ell(\xi_s) Y_s^t ds = \Pi_x \int_0^\tau dt \rho(\xi_t)(1 - Y_0^t).$$

That implies (1.7). The uniqueness of a solution of (1.7) can be proved in the same way as it was done in [**D**] for (1.6). [It follows also from [**D**], Lemma 8.2.2.]

1.3. ***h*-transform.** Let ξ be a diffusion in E. Denote by $\mathcal{F}_{\leq t}^\xi$ the σ-algebra generated by the sets $\{\xi_s \in B, s < \tau_E\}$ with $s \leq t, B \in \mathcal{B}(E)$. Denote $\mathcal{F}^\xi$ the minimal σ-algebra which contains all $\mathcal{F}_{\leq t}^\xi$. Let $p_t(x, y)$ be the transition density of ξ and let $h \in \mathcal{H}$. To every $x \in E$ there corresponds a finite measure Π_x^h on $\mathcal{F}^\xi$ such that, for all $0 < t_1 < \cdots < t_n$ and every Borel sets $B_1, \dots, B_n$,

$$(1.9) \quad \Pi_x^h \{\xi_{t_1} \in B_1, \dots, \xi_{t_n} \in B_n\}$$
$$= \int_{B_1} dz_1 \dots \int_{B_n} dz_n \; p_{t_1}(x, z_1) p_{t_2 - t_1}(z_1, z_2) \dots p_{t_n - t_{n-1}}(z_{n-1}, z_n) h(z_n).$$

Note that $\Pi_x^h(\Omega) = h(x)$ and therefore $\hat{\Pi}_x^h = \Pi_x^h / h(x)$ is a probability measure. $(\xi_t, \hat{\Pi}_x^h)$ is a strong Markov process with continuous paths and with the transition density

$$p_t^h(x, y) = \frac{1}{h(x)} p_t(x, y) h(y). \tag{1.10}$$

We use the following properties of h-transforms.

1.3.A. If $Y \in \mathcal{F}_{\leq t}^\xi$, then

$$\Pi_x^h 1_{t < \tau_E} Y = \Pi_x 1_{t < \tau_E} Y h(\xi_t).$$

[This follows immediately from (1.9).]

1.3.B. [[**D**], Lemma 7.3.1.] For every stopping time τ and every pre-τ positive Y,

$$\Pi_x^h Y 1_{\tau < \tau_E} = \Pi_x Y h(\xi_\tau) 1_{\tau < \tau_E}.$$

1.4. Conditional *L*-diffusion. We put $\Pi_x^\nu = \Pi_x^{h_\nu}$ where h_ν is given by **1**.(1.3). For every $x \in E, y \in \partial E$, we put $\Pi_x^y = \Pi_x^{\delta_y} = \Pi_x^h$ and $\hat{\Pi}_x^y = \hat{\Pi}_x^h$ where $h(\cdot) = k_E(\cdot, y)$. Let $Z = \xi_{\tau_E} 1_{\tau_E < \infty}$. It follows from the definition of the Poisson operator and (1.1) that, for every $\varphi \in \mathcal{B}(\partial E)$,

$$\Pi_x \varphi(Z) = \int_{\partial E} k_E(x, z) \varphi(z) \gamma(dz). \tag{1.11}$$

Therefore

$$\Pi_x k_E(y,Z)\varphi(Z) = \int_{\partial E} k_E(x,z)k_E(y,z)\varphi(z)\gamma(dz) \tag{1.12}$$

is symmetric in x, y.

LEMMA 1.1. [1] *For every $Y \in \mathcal{F}^\xi$ and every $f \in \mathcal{B}(\partial E)$,*

$$\Pi_x Y f(Z) = \Pi_x f(Z)\hat{\Pi}_x^Z Y. \tag{1.13}$$

PROOF. It is sufficient to prove (1.13) for $Y = Y' 1_{t<\tau_E}$ where $Y' \in \mathcal{F}^\xi_{\le t}$. By 1.3.A,

$$\hat{\Pi}_x^z Y = k_E(x,z)^{-1}\Pi_x^z Y = k_E(x,z)^{-1}\Pi_x Y k_E(\xi_t, z).$$

Therefore the right part in (1.13) can be interpreted as

$$\int_{\Omega'} \Pi_x(d\omega') f(Z(\omega'))k_E(x, Z(\omega'))^{-1} \int_\Omega \Pi_x(d\omega)Y(\omega)k_E(\xi_t(\omega), Z(\omega')).$$

Fubini's theorem and (1.12) (applied to $\varphi(z) = f(z)k_E(x,z)^{-1}$) yield that this expression is equal to

$$\int_\Omega \Pi_x(d\omega)Y(\omega)\int_{\Omega'} \Pi_x(d\omega')f(Z(\omega'))k_E(\xi_t(\omega), Z(\omega'))k_E(x, Z(\omega'))^{-1}$$
$$= \int_\Omega \Pi_x(d\omega)Y(\omega)\int_{\partial E} f(z)k_E(\xi_t(\omega), z)\gamma(dz).$$

By (1.11), the right side is equal to

$$\Pi_x Y \Pi_{\xi_t} f(Z) = \Pi_x Y' 1_{t<\tau_E}\Pi_{\xi_t} f(Z).$$

Since $Y' \in \mathcal{F}^\xi_{\le t}$, the Markov property of ξ implies that this is equal to the left side in (1.13). □

Suppose that $\xi = (\xi_t, \Pi_x)$ is an L diffusion in E and let $\tilde{L}$ be the restriction of L to an open subset D of E. An $\tilde{L}$-diffusion $\tilde{\xi} = (\tilde{\xi}_t, \tilde{\Pi}_x)$ can be obtained as the part of ξ in D defined by the formulae

$$\tilde{\xi}_t = \xi_t \quad \text{for } 0 \le t < \tau_D,$$
$$\tilde{\Pi}_x = \Pi_x \quad \text{for } x \in D.$$

Notation $\tilde{\Pi}_x^y$ refers to the diffusion $\tilde{\xi}$ started at $x \in D$ and conditioned to exit from D at $y \in \partial D$. A relation between $\tilde{\Pi}_x^y$ and Π_x^y is established by the following lemma.

LEMMA 1.2. *Suppose that $D \subset E$ are smooth open sets. For every $x \in D, y \in \partial D \cap \partial E$, and $Y \in \mathcal{F}^{\tilde{\xi}}$,*

$$\tilde{\Pi}_x^y Y = \Pi_x^y\{\tau_D = \tau_E, Y\}. \tag{1.14}$$

[1] Property (1.13) means that $\hat{\Pi}_x^z$ can be interpreted as the conditional probability distribution given that the diffusion started from x exits from E at point z.

PROOF. It is sufficient to prove (1.14) for $Y = \tilde{Y} 1_{t<\tau_D}$ where $\tilde{Y} \in \mathcal{F}^{\tilde{\xi}}_{\le t}$. By 1.3.A, 1.1.C, 1.3.B and Markov property of ξ,

$$\begin{aligned}\tilde{\Pi}^y_x Y = \Pi_x Y k_D(\tilde{\xi}_t, y) &= \Pi_x Y[k_E(\xi_t, y) - \Pi_{\xi_t} 1_{\tau_D<\tau_E} k_E(\xi_{\tau_D}, y)] \\ &= \Pi_x Y k_E(\xi_t, y) - \Pi_x Y 1_{\tau_D<\tau_E} k_E(\xi_{\tau_D}, y) = \Pi^y_x Y - \Pi^y_x Y 1_{\tau_D<\tau_E}\end{aligned}$$

which implies (1.14). □

COROLLARY 1.1. *If*

$$F_t = \exp\Big[-\int_0^t a(\xi_s)\, ds\Big] \tag{1.15}$$

where a is a positive continuous function on $[0,\infty)$, then, for $y \in \partial D \cap \partial E$,

$$\tilde{\Pi}^y_x F_{\tau_D} = \Pi^y_x\{\tau_D = \tau_E, F_{\tau_E}\}. \tag{1.16}$$

Since $F_{\tau_D} \in \mathcal{F}^{\tilde{\xi}}$, this follows from (1.14).

2. Superprocesses

2.1. Branching exit Markov systems. A *random measure* on a measurable space $(E, \mathcal{B})$ is a pair (X, P) where $X(\omega, B)$ is a kernel from an auxiliary measurable space $(\Omega, \mathcal{F})$ to $(E, \mathcal{B})$ and P is a measure on $\mathcal{F}$. We assume that E is a metric space and $\mathcal{B}$ is the class of all Borel subsets of E.

Suppose that:

(i) $\mathbb{O}$ is the class of all open subsets of E;

(ii) to every $D \in \mathbb{O}$ and every $\mu \in \mathcal{M}(E)$ there corresponds a random measure (X_D, P_μ) on $(E, \mathcal{B})$.

Denote by $\mathcal{Z}$ the class of functions

$$Z = \sum_1^n \langle f_i, X_{D_i}\rangle \tag{2.1}$$

where $D_i \in \mathbb{O}$ and $f_i \in \mathcal{B}$ and put $Y \in \mathcal{Y}$ if $Y = e^{-Z}$ where $Z \in \mathcal{Z}$. We say that X is a *branching exit Markov [BEM] system* [2] if $X_D \in \mathcal{M}(E)$ for all $D \in \mathbb{O}$ and if:

2.1.A. For every $Y \in \mathcal{Y}$ and every $\mu \in \mathcal{M}(E)$,

$$P_\mu Y = e^{-\langle u, \mu\rangle} \tag{2.2}$$

where

$$u(y) = -\log P_y Y \tag{2.3}$$

and $P_y = P_{\delta_y}$.

2.1.B. For all $\mu \in \mathcal{M}(E)$ and $D \in \mathbb{O}$,

$$P_\mu\{X_D(D) = 0\} = 1.$$

[2] This concept in a more general setting is introduced in **[D]**, Chapter 3.

2.1.C. If $\mu \in \mathcal{M}(E)$ and $\mu(D) = 0$, then

$$P_\mu\{X_D = \mu\} = 1.$$

2.1.D. [Markov property.] Suppose that $Y \geq 0$ is measurable with respect to the σ-algebra $\mathcal{F}_{\subset D}$ generated by $X_{D'}, D' \subset D$ and $Z \geq 0$ is measurable with respect to the σ-algebra $\mathcal{F}_{\supset D}$ generated by $X_{D''}, D'' \supset D$. Then

$$P_\mu(YZ) = P_\mu(Y P_{X_D} Z). \tag{2.4}$$

Condition 2.1.A (we call it the *continuous branching property*) implies that

$$P_\mu Y = \prod P_{\mu_n} Y$$

for all $Y \in \mathcal{Y}$ if $\mu_n, n = 1, 2, \dots$ and $\mu = \sum \mu_n$ belong to $\mathcal{M}(E)$.

There is a degree of freedom in the choice of the auxiliary space $(\Omega, \mathcal{F})$. We say that a system (X_D, P_μ) is *canonical* if Ω consists of all $\mathcal{M}$-valued functions ω on $\mathbb{O}$, if $X_D(\omega, B) = \omega(D, B)$ and if $\mathcal{F}$ is the σ-algebra generated by the sets $\{\omega : \omega(D, B) < c\}$ with $D \in \mathbb{O}, B \in \mathcal{B}, c \in \mathbb{R}$.

We will use the following implications of conditions 2.1.A–2.1.D:

2.1.E. [[**D**], 3.4.2.D] If $D' \subset D''$ belong to $\mathbb{O}$ and if $B \in \mathcal{B}$ is contained in the complement of D'', then $X_{D'}(B) \leq X_{D''}(B)$ P_x-a.s. for all $x \in E$.

2.1.F. If $\mu = 0$, then $P_\mu\{Z = 0\} = 1$ for every $Z \in \mathcal{Z}$.

This follows from 2.1.A.

2.1.G. If $D \subset \tilde{D}$, then

$$\{X_D = 0\} \subset \{X_{\tilde{D}} = 0\} \quad P_\mu\text{-a.s.} \tag{2.5}$$

Indeed, by 2.1.D and 2.1.F,

$$P_\mu\{X_D = 0, X_{\tilde{D}} \neq 0\} = P_\mu\{X_D = 0, P_{X_D}[X_{\tilde{D}} = 0]\} = 0.$$

2.2. Definition and existence of superprocesses. Suppose that $\xi = (\xi_t, \Pi_x)$ is a time-homogeneous right continuous strong Markov process in a metric space E. We say that a BEM system $X = (X_D, P_\mu), D \in \mathbb{O}, \mu \in \mathcal{M}(E)$ is a (ξ, ψ)-*superprocess* if, for every $f \in b\mathcal{B}(E)$ and every $D \in \mathbb{O}$,

$$V_D f(x) = -\log P_x e^{-\langle f, X_D \rangle} \tag{2.6}$$

where $P_x = P_{\delta_x}$ and V_D are operators introduced in Section **2**.2. By 2.1.A,

$$P_\mu e^{-\langle f, X_D \rangle} = e^{-\langle V_D(f), \mu \rangle} \quad \text{for all } \mu \in \mathcal{M}(E). \tag{2.7}$$

The existence of a (ξ, ψ)-superprocesses is proved in [**D**],Theorem 4.2.1 for

$$\psi(x; u) = b(x)u^2 + \int_0^\infty (e^{-tu} - 1 + tu) N(x; dt) \tag{2.8}$$

under broad conditions on a positive Borel function $b(x)$ and a kernel N from E to $\mathbb{R}_+$. It is sufficient to assume that:

$$b(x), \int_1^\infty tN(x;dt) \quad \text{and} \quad \int_0^1 t^2 N(x;dt) \quad \text{are bounded.} \tag{2.9}$$

An important special case is the function

$$\psi(x,u) = \ell(x)u^\alpha, 1 < \alpha \le 2 \tag{2.10}$$

corresponding to $b = 0$ and

$$N(x,dt) = \tilde{\ell}(x)t^{-1-\alpha}dt$$

where

$$\tilde{\ell}(x) = \ell(x)[\int_0^\infty (e^{-\lambda} - 1 + \lambda)\lambda^{-1-\alpha}d\lambda]^{-1}.$$

Condition (2.9) holds if $\ell(x)$ is bounded.

Under the condition (2.9), the derivatives $\psi_r(x,u) = \frac{\partial^r \psi(x,u)}{\partial u^r}$ exist for $u > 0$ for all r. Moreover,

$$\begin{aligned} \psi_1(x,u) &= 2bu + \int_0^\infty t(1-e^{-tu})N(x,dt), \\ \psi_2(x,u) &= 2b + \int_0^\infty t^2 e^{-tu} N(x,dt), \\ (-1)^r \psi_r(x,u) &= \int_0^\infty t^r e^{-tu} N(x,dt) \quad \text{for } 2 < r \le n. \end{aligned} \tag{2.11}$$

Put $\mu \in \mathcal{M}_c(E)$ if $\mu \in \mathcal{M}(U)$ for some $U \Subset E$. In this book we consider only superprocesses corresponding to continuous processes ξ. This implies $\xi_{\tau_D} \in \partial D$ Π_x-a.s. for every $x \in D$. It follows from 1.1.A and **2**.(2.1)that

2.2.A. For every $\mu \in \mathcal{M}_c(D)$, X_D is supported by ∂D P_μ-a.s.

The condition **1**.4.3.B implies

2.2.B. [[**D**], Lemma 4.4.1]

$$P_\mu \langle f, X_D \rangle = \langle K_D f, \mu \rangle \tag{2.12}$$

for every open set $D \subset E$, every $f \in \mathcal{B}(E)$ and every $\mu \in \mathcal{M}(E)$.

2.3. Random closed sets. Suppose $(\Omega, \mathcal{F})$ is a measurable space, E is a locally compact metrizable space and $\omega \to F(\omega)$ is a map from Ω to the collection of all closed subsets of E. Let P be a probability measure on $(\Omega, \mathcal{F})$. We say that (F, P) is a *random closed set (r.c.s.)* if, for every open set U in E,

$$\{\omega : F(\omega) \cap U = \emptyset\} \in \mathcal{F}^P \tag{2.13}$$

where $\mathcal{F}^P$ is the completion of $\mathcal{F}$ relative to P. Two r.c.s. (F, P) and $(\tilde{F}, P)$ are equivalent if $P\{F = \tilde{F}\} = 1$.

Suppose $(F_a, P), a \in A$ is a family of r.c.s. We say that a r.c.s. (F, P) is an *envelope* of (F_a, P) if:

(a) $F_a \subset F$ P-a.s. for every $a \in A$.

(b) If (a) holds for $\tilde{F}$, then $F \subset \tilde{F}$ P-a.s.

An envelope exists for every countable family. For an uncountable family, it exists under certain separability assumptions. Note that the envelope is determined uniquely up to equivalence and that it does not change if every r.c.s. (F_a, P) is replaced by an equivalent set.

Suppose that (M, P) is a random measure on E.The support $\mathbb{S}$ of M satisfies condition

$$\{\mathbb{S} \cap U = \emptyset\} = \{M(U) = 0\} \in \mathcal{F} \tag{2.14}$$

for every open subset U of E and therefore $\mathbb{S}(\omega)$ is a r.c.s.

An important class of capacities related to random closed sets has been studied in the original memoir of Choquet [Ch54]. Let (F, P) be a random closed set in E. Put

$$\Lambda_B = \{\omega : F(\omega) \cap B \neq \emptyset\}. \tag{2.15}$$

The definition of a random closed set implies Λ_B belongs to the completion $\mathcal{F}^P$ of $\mathcal{F}$ for all B in $\mathcal{K}$.

Note that

$$\begin{gathered} \Lambda_A \subset \Lambda_B \quad \text{if } A \subset B, \\ \Lambda_{A \cup B} = \Lambda_A \cup \Lambda_B, \quad \Lambda_{A \cap B} \subset \Lambda_A \cap \Lambda_B, \\ \Lambda_{B_n} \uparrow \Lambda_B \quad \text{if } B_n \uparrow B, \\ \Lambda_{K_n} \downarrow \Lambda_K \quad \text{if } K_n \downarrow K \quad \text{and } K_n \in \mathcal{K}. \end{gathered}$$

Therefore the function

$$\mathrm{Cap}(K) = P(\Lambda_K), \quad K \in \mathcal{K} \tag{2.16}$$

satisfies conditions **2**.4.A–**2**.4.D and it can be continued to a capacity on E. Clearly, $\Lambda_O \in \mathcal{F}^P$ for all $O \in \mathbb{O}$. It follows from **2**.4.B that $\mathrm{Cap}(O) = P(\Lambda_O)$ for all open O. Suppose that B is a Borel set. By **2**.(4.1), there exist $K_n \in \mathcal{K}$ and $O_n \in \mathcal{O}$ such that $K_n \subset B \subset O_n$ and $\mathrm{Cap}(O_n) - \mathrm{Cap}(K_n) < 1/n$. Since $\Lambda_{K_n} \subset \Lambda_B \subset \Lambda_{O_n})$ and since $P(\Lambda_{O_n}) - P(\Lambda_{K_n}) = \mathrm{Cap}(O_n) - \mathrm{Cap}(K_n) < 1/n$, we conclude that $\Lambda_B \in \mathcal{F}^P$ and

$$\mathrm{Cap}(B) = P(\Lambda_B). \tag{2.17}$$

2.4. Range of a superprocess. We consider a (ξ, ψ)-superprocess X corresponding to a continuous strong Markov process ξ. Let $\mathcal{F}$ be the σ-algebra in Ω generated by $X_O(U)$ corresponding to all open sets $O \subset E, U \subset \mathbb{R}^d$. The support $\mathbb{S}_O$ of X_O is a closed subset of $\bar{E}$. To every open set O and every $\mu \in \mathcal{M}(E)$ there corresponds a r.c.s. $(\mathbb{S}_O, P_\mu)$ in $\bar{E}$ (defined up to equivalence). By [**D**], Theorem 4.5.1, for every E and every μ, there exists an envelope $(\mathcal{R}_E, P_\mu)$ of the family $(\mathbb{S}_O, P_\mu), O \subset E$. We call it the *range of* X in E.

The random set $\mathcal{R}_E$ can be constructed as follows. Consider a sequence of open subsets $O_1, \dots, O_n, \dots$ of E such that for every open set $O \subset E$ there exists a subsequence O_{n_k} exhausting O. [3] Put

$$M = \sum \frac{1}{a_n 2^n} X_{O_n} \tag{2.18}$$

where $a_n = \langle 1, X_{O_n} \rangle \vee 1$ and define $\mathcal{R}_E$ as the support of the measure M.

We state an important relation between exit measures and the range.

2.4.A. [[**D**], Theorem 4.5.3] Suppose K is a compact subset of ∂E and let $D_n = \{x \in E : d(x, K) > 1/n\}$. Then

$$\{X_{D_n}(E) = 0\} \uparrow \{\mathcal{R}_E \cap K = \emptyset\} \quad P_x\text{-a.s.} \tag{2.19}$$

for all $x \in E$.

3. Superdiffusions

3.1. Definition. If ξ is an L-diffusion, then the (ξ, ψ)-superprocess is called an (L, ψ)-*superdiffusion*. If D is a bounded smooth domain and if f is continuous, then, under broad assumptions on ψ, the integral equation **2**.(2.1) is equivalent to the differential equation $Lu = \psi(u)$ with the boundary condition $u = f$.

3.2. Family $\langle u, X_D \rangle$, $u \in \mathcal{U}^*$.

THEOREM 3.1. [4] *Suppose D_n is a sequence exhausting E and let $\mu \in \mathcal{M}_c(E)$. If $u \in \mathcal{U}^-(E)$ ($u \in \mathcal{U}^+(E)$) then $Y_n = e^{-\langle u, X_{D_n} \rangle}$ is a submartingale (supermartingale) relative to $(\mathcal{F}_{\subset D_n}, P_\mu)$. For every $u \in \mathcal{U}^*$, there exists, P_μ-a.s., $\lim \langle u, X_{D_n} \rangle = Z$.*

PROOF. By the Markov property 2.1.D, for every $A \in \mathcal{F}_{\subset D_n}$,

$$P_\mu 1_A Y_{n+1} = P_\mu 1_A P_{X_{D_n}} Y_{n+1}.$$

Therefore the first statement of the theorem follows from the definition of $\mathcal{U}^-(E)$ and $\mathcal{U}^+(E)$. The second statement follows from the first one by a well-known convergence theorem for bounded submartingales and supermartingales (see, e.g., [**D**], Appendix A, 4.3.A). □

3.3. Stochastic boundary values. Suppose that $u \in \mathcal{B}(E)$ and, for every sequence D_n exhausting E,

$$\lim \langle u, X_{D_n} \rangle = Z_u \quad P_\mu\text{-a.s. for all } \mu \in \mathcal{M}_c(E). \tag{3.1}$$

Then we say that Z_u is a *stochastic boundary value of* u and we write $Z_u = \text{SBV}(u)$.

[3]For instance, take a countable everywhere dense subset Λ of E. Consider all balls contained in E centered at points of Λ with rational radii and enumerate all finite unions of these balls.

[4]Cf. Theorem 9.1.1 in [**D**].

Clearly, Z is defined by (3.1) uniquely up to equivalence. [We say that Z_1 and Z_2 are equivalent if $Z_1 = Z_2$ P_μ-a.s. for every $\mu \in \mathcal{M}_c(E)$.] [5] We call u the *log-potential* of Z and we write $u = \mathrm{LPT}(Z)$ if

$$u(x) = -\log P_x e^{-Z} \tag{3.2}$$

THEOREM 3.2 ([**D**], Theorem 9.1.1). *The stochastic boundary value exists for every $u \in \mathcal{U}^-(E)$ and every $u \in \mathcal{U}^+(E)$. If $Z_u = \mathrm{SBV}(u)$ exists, then $u \in \mathcal{D}(\pi)$ and , for every $\mu \in \mathcal{M}_c$,*

$$P_\mu e^{-Z_u} = e^{-\langle \pi(u), \mu\rangle}. \tag{3.3}$$

In particular, if $u \in \mathcal{U}(E)$, then

$$u(x) = -\log P_x e^{-Z_u} \quad \textit{for every } x \in E. \tag{3.4}$$

PROOF. Let D_n exhaust E. By (2.7) and (3.1),

$$e^{-\langle V_{D_n}(u), \mu\rangle} = P_\mu e^{-\langle u, X_{D_n}\rangle} \to P_\mu e^{-Z_u}. \tag{3.5}$$

Hence, $\lim V_{D_n}(u)(x)$ exists for every $x \in E$, $u \in \mathcal{D}(\pi)$. By **2**.2.2.E, for every $D \Subset E$, the family of functions $V_{D_n}(u), D_n \supset D$ are uniformly bounded and therefore $\langle V_{D_n}(u), \mu\rangle \to \langle \pi(u), \mu\rangle$. We get (3.3) by a passage to the limit in (3.5).

(3.4) follows because $\pi(u) = u$ for $u \in \mathcal{U}(E)$ by **2**.2.2.C. □

Here are more properties of stochastic boundary values.

3.3.A. If $\mathrm{SBV}(u)$ exists, then it is equal to $\mathrm{SBV}(\pi(u))$.

PROOF. Let D_n exhaust E and let $\mu \in \mathcal{M}_c(E)$. By (3.3) and the Markov property,

$$e^{-\langle \pi(u), X_{D_n}\rangle} = P_{X_{D_n}} e^{-Z_u} = P_\mu\{e^{-Z_u} | \mathcal{F}_{\subset D_n}\} \to e^{-Z_u} \quad P_\mu\text{–a.s.}$$

Hence, $\langle \pi(u), X_{D_n}\rangle \to Z_u$ P_μ-a.s. □

3.3.B. If $\mathrm{SBV}(u) = Z_u$ and $\mathrm{SBV}(v) = Z_v$ exist, then $\mathrm{SBV}(u + v)$ exists and

$$\mathrm{SBV}(u + v) = \mathrm{SBV}(u) + \mathrm{SBV}(v) = \mathrm{SBV}(u \oplus v).$$

The first equation follows immediately from the definition of SBV. It implies that the second one follows by 3.3.A.

LEMMA 3.1. *If, $u \geq v \in \mathcal{U}(E)$, then*

$$(u \ominus v) \oplus v = u. \tag{3.6}$$

[5] It is possible that Z_1 and Z_2 are equivalent but $P_\mu\{Z_1 \neq Z_2\} > 0$ for some $\mu \in \mathcal{M}(E)$.

PROOF. If $u \geq v \in \mathcal{U}(E)$, then, by **2**.2.3.D and **2**.3.F, $u - v \in \mathcal{U}^+$ and $u \ominus v = \pi(u - v)$. Therefore, by 3.3.A and 3.3.B,

$$Z_{u\ominus v} = Z_{u-v} = Z_u - Z_v \quad P_x\text{-a.s. on } \{Z_v < \infty\}.$$

Hence,

$$Z_u = Z_v + Z_{u\ominus v} \quad P_x\text{-a.s. on } \{Z_v < \infty\}. \tag{3.7}$$

Since $Z_u \geq Z_v$ P_x-a.s, this equation holds also on $\{Z_\nu = \infty\}$. Since $u \ominus v$ and v belong to $\mathcal{U}(E)$, $u \ominus v + v \in \mathcal{U}^-(E)$ by **2**.3.D and, by 3.3.A and 3.3.B,

$$Z_{(u\ominus v)\oplus v} = Z_{(u\ominus v)+v} = Z_{(u\ominus v)} + Z_v = Z_u.$$

Because of (3.4), this implies (3.6). □

3.4. Linear boundary functionals. Denote by $\mathcal{F}_{\subset E-}$ the minimal σ-algebra which contains $\mathcal{F}_{\subset D}$ for all $D \Subset E$ and by $\mathcal{F}_{\supset E-}$ the intersection of $\mathcal{F}_{\supset D}$ over all $D \Subset E$. Note that, if D_n is a sequence exhausting E, then $\mathcal{F}_{\subset E-}$ is generated by the union of $\mathcal{F}_{\subset D_n}$ and $\mathcal{F}_{\supset E-}$ is the intersection of $\mathcal{F}_{\supset D_n}$.

We define the germ σ-algebra on the boundary $\mathcal{F}_\partial$ as the completion of the σ-algebra $\mathcal{F}_{\subset E-} \cap \mathcal{F}_{\supset E-}$ with respect to the family of measures $P_\mu, \mu \in \mathcal{M}_c(E)$. We say that a positive function Z is a *linear boundary functional* [6] if

3.4.1. Z is $\mathcal{F}_\partial$-measurable.

3.4.2. For all $\mu \in \mathcal{M}_c(E)$,

$$-\log P_\mu e^{-Z} = \int [-\log P_x e^{-Z}] \mu(dx).$$

3.4.3. $\quad P_x\{Z < \infty\} > 0 \quad$ for all $x \in E$.

We denote by $\mathfrak{Z}$ the set of all such functionals (two functionals that coincide P_μ-a.s. for all $\mu \in \mathcal{M}_c(E)$ are identified).

THEOREM 3.3. *[[**D**], Theorem 9.1.2] The stochastic boundary value Z of any $u \in \mathcal{U}^-(E) \cup \mathcal{U}^+(E)$ belongs to $\mathfrak{Z}$. Let $Z \in \mathfrak{Z}$. Then the log-potential u of Z belongs to $\mathcal{U}(E)$ and Z is the stochastic boundary value of u.*

According to Theorem 9.1.3 in [**D**],

3.4.A. If $Z_1, Z_2 \in \mathfrak{Z}$, then $Z_1 + Z_2 \in \mathfrak{Z}$ and

$$\mathrm{LPT}(Z_1 + Z_2) \leq \mathrm{LPT}(Z_1) + \mathrm{LPT}(Z_2). \tag{3.8}$$

3.4.B. If $Z_1, \dots, Z_n, \dots \in \mathfrak{Z}$ and if $Z_n \to Z$ P_μ-a.s for all $\mu \in \mathcal{M}_c(E)$, then $Z \in \mathfrak{Z}$.

It follows from [**D**], 9.2.2.B that:

[6]The word "boundary" refers to condition 3.4.1 and the word "linear" refers to 3.4.2.

3.4.C. If $Z \in \mathfrak{Z}$ and if $h(x) = P_x Z$ is finite at some point $x \in E$, then $h \in \mathcal{H}_1(E)$ and $u(x) = -\log P_x e^{-Z}$ is a moderate solution.

3.5. On solutions w_Γ. These solutions can be expressed in terms of the range of the (L, ψ)-superdiffusion by the formula

$$w_\Gamma(x) = -\log P_x\{\mathcal{R}_E \cap \Gamma = \emptyset\}. \tag{3.9}$$

[See [**D**], Theorem 10.3.1.] By taking $\Gamma = \partial E$, we get the maximal element of $\mathcal{U}(E)$

$$w(x) = -\log P_x\{\mathcal{R}_E \subset E\}. \tag{3.10}$$

This solution can also be expressed through the range $\mathcal{R}$ in the entire space $\mathbb{R}^d$ (assuming that ξ is defined in $\mathbb{R}^d$)

$$w(x) = -\log P_x\{\mathcal{R} \subset E\}. \tag{3.11}$$

Indeed, if $x \in E$, then, P_x-a.s. X_E is concentrated on $\mathcal{R}_E \cap \partial E$. If $\mathcal{R}_E \subset E$, then $P_x\{X_E = 0\} = 1$ and, by 2.1.G, $X_O = 0$ P_x-a.s. for all $O \supset E$. Hence, the envelope of $\mathbb{S}_O$, $O \subset \mathbb{R}^d$ coincide, P_x-a.s. on $\mathcal{R}_E \subset E$, with the envelope of $\mathbb{S}_O$, $O \subset E$.

We need the following properties of w_Γ:

3.5.A. w_Γ is the log-potential of

$$Z_\Gamma = \begin{cases} 0 & \text{if } \mathcal{R}_E \cap \Gamma = \emptyset, \\ \infty & \text{if } \mathcal{R}_E \cap \Gamma \neq \emptyset \end{cases}$$

and

$$\mathrm{SBV}(w_\Gamma) = Z_\Gamma.$$

[See Theorem 3.3 and [**D**], Remark 1.2, p. 133.]

3.5.B. [[**D**], 10.(3.1) and 10.(3.6)] For every Borel set $\Gamma \subset \partial E$, $w_\Gamma(x)$ is equal to the infimum of $w_O(x)$ over all open subsets $O \supset \Gamma$ of ∂E.

3.5.C. [[**D**], 10.1.3.A and 10.1.3.E] If $\Gamma \subset A \cup B$, then $w_\Gamma \leq w_A + w_B$.

3.6. Stochastic boundary value of h_ν and u_ν. Recall that to every $\nu \in \mathcal{M}(\partial E)$ there corresponds a harmonic function

$$h_\nu(x) = \int_{\partial E} k_E(x, y)\nu(dy)$$

[cf. **1**.(1.3)] and a solution u_ν [the maximal element of $\mathcal{U}(E)$ dominated by h_ν]. A linear boundary functional

$$Z_\nu = \mathrm{SBV}(h_\nu) \tag{3.12}$$

has the following propertries:

3.6.A. [[**D**], 9.(2.1)] For all $x \in E$,

$$P_x Z_\nu \leq h_\nu(x).$$

3.6.B. [[**D**].9.2.2.B] If $\nu \in \mathcal{N}_1^E$, then, for all $x \in E$, $P_x Z_\nu = h_\nu(x)$ and

$$u_\nu + G_E \psi(u_\nu) = h_\nu.$$

3.6.C. For every $\nu \in \mathcal{N}_1^E$, $\mathrm{SBV}(h_\nu) = \mathrm{SBV}(u_\nu)$.

Indeed, $\mathrm{SBV}(h_\nu) = \mathrm{SBV}(\pi(h_\nu))$ by 3.3.A and $\pi(h_\nu) = u_\nu$ by **2**.3.A.

A σ-moderate solution u_ν is defined by Lemma **2**.2.1 for every $\nu \in \mathcal{N}_0^E$. We put $Z_\nu = \mathrm{SBV}(u_\nu)$ which is consistent with (3.12) because $\mathcal{N}_0^E \cap \mathcal{M}(\partial E) = \mathcal{N}_1^E$ by **2**.2.4.A and $\mathrm{SBV}(u_\nu) = \mathrm{SBV}(h_\nu)$ by 3.6.C.

It follows from (3.4) that

$$u_\nu(x) = -\log P_x e^{-Z_\nu} \quad \text{for all } \nu \in \mathcal{N}_0^E. \tag{3.13}$$

Clearly, this implies

$$u_{\infty\cdot\nu}(x) = -\log P_x\{Z_\nu = 0\}. \tag{3.14}$$

LEMMA 3.2. *For every $\lambda, \nu \in \mathcal{N}_1^E$,*

$$u_\lambda \oplus u_\nu = u_{\lambda+\nu}. \tag{3.15}$$

PROOF. By **2**.2.3.D, $u_\lambda + u_\nu \in \mathcal{U}^-(E)$ and so, by 3.3.A, $\mathrm{SBV}(\pi(u_\lambda + u_\nu)) = \mathrm{SBV}(u_\lambda + u_\nu)$. Since $\pi(u_\lambda + u_\nu) = u_\lambda \oplus u_\nu$, we get $\mathrm{SBV}(u_\lambda \oplus u_\nu) = \mathrm{SBV}(u_\lambda + u_\nu)$. By 3.6.C, the right side is equal to $\mathrm{SBV}(u_{\lambda+\nu})$, and (3.15) follows from (3.13). $\square$

3.7. Relation between the range and Z_ν.

THEOREM 3.4. *Suppose that $\nu \in \mathcal{N}_1^E$ is concentrated on a Borel set $\Gamma \subset \partial E$. Then*

$$P_x\{\mathcal{R}_E \cap \Gamma = \emptyset, Z_\nu \neq 0\} = 0. \tag{3.16}$$

PROOF. Let D_n exhaust E. We claim that

$$Z_\nu = \lim \langle u_\nu, X_{D_n}\rangle \quad P_x\text{-a.s.} \tag{3.17}$$

Indeed,

$$P_x e^{-\langle u_\nu, X_{D_n}\rangle} = e^{-u_\nu(x)}$$

by 2.3.F. By passing to the limit, we get

$$P_x e^{-Z} = e^{-u_\nu(x)}$$

where

$$Z = \lim \langle u_\nu, X_{D_n}\rangle.$$

This means $u_\nu = \mathrm{LPT}\, Z$. By Theorem **3**.3.3, $Z = \mathrm{SBV}(u_\nu) = Z_\nu$.

Since $u_\nu \le h_\nu = 0$ on $\partial E \setminus \Gamma$, we have

$$\langle X_{D_n}(E) = 0\} = \{\langle u_\nu, X_{D_n}\rangle = 0\}$$

and, by 2.4.A,

$$P_x\{\mathcal{R}_E \cap \Gamma = \emptyset, Z_\nu \neq 0\} = \lim P_x\{\langle u_\nu, X_{D_n}\rangle = 0, Z_\nu \neq 0\rangle\} = 0.$$

$\square$

3.8. $\mathcal{R}_E$-polar sets and class $\mathcal{N}_1^E$. We say that a subset Γ of ∂E is $\mathcal{R}_E$-polar if $P_x\{\mathcal{R}_E \cap \Gamma = \emptyset\} = 1$ for all $x \in E$.

THEOREM 3.5. *Class $\mathcal{N}_1^E$ associated with the equation*

$$\Delta u = u^\alpha, 1 < \alpha \leq 2$$

in a bounded smooth domain E consists of all finite measures ν on ∂E charging no $\mathcal{R}_E$-polar set.

This follows from proposition 10.1.4.C, Theorem 13.0.1 and Theorem 12.1.2 in [**D**].

4. Notes

In this chapter we summarize the theory of superdiffusion presented in [**D**]. Our first publication [Dy91a] on this subject was inspired by a paper [Wa68] of S. Watanabe where a superprocess corresponding to $\psi(x, u) = b(x)u^2$ has been constructed by a passage to the limit from a branching particle system. [Another approach to supeprocesses via Ito's stochastic calculus was initiated by Dawson in [Da75].] Till the beginning of the 1990s superprocesses were interpreted as measure-valued Markov processes X_t. However, for applications to partial differential equations it is not sufficient to deal with the mass distribution at fixed times t. A model of superdiffusions as systems of exit measures from open sets was developed in [Dy91a], [Dy92] and [Dy93]. For these systems a Markov property and a continuous branching property were established and applied to boundary value problems for semilinear equations. In [**D**] the entire theory of superdiffusion was deduced from these properties.

A mass distribution at fixed time t can be interpreted as the exit measure from the time-space domain $(-\infty, t) \times \mathbb{R}^d$. To cover these distributions, we consider in Part I of [**D**] systems of exit measures from all time-space open sets and we apply these systems to parabolic semilinear equations. In Part II, the results for elliptic equations are deduced from their parabolic counterpart. In the present book we consider only the elliptic case and therefore we can restrict ourselves by exit measures from subsets of $\mathbb{R}^d$. Since the technique needed in parabolic case is more complicated and since the most results are easier to formulate in the elliptic case, there is a certain advantage in reading the first three chapters of the present book before a systematic reading of [**D**].

More information about the literature on superprocesses and on related topics can be found in Notes in [**D**].

CHAPTER 4

$\mathbb{N}$-measures

$\mathbb{N}$-measures appeared, first, as excursion measures of the Brownian snake – a path-valued Markov process introduced by Le Gall and used by him and his school for investigating the equation $\Delta u = u^2$. In particular, they play a key role in Mselati's dissertation. In Le Gall's theory, measures $\mathbb{N}_x$ are defined on the space of continuous paths. We define their analog in the framework of superprocesses (and general branching exit Markov systems) on the same space Ω as measures P_μ.

To illustrate the role of these measures, we consider probabilistic solutions of the equation $Lu = \psi(u)$ in a bounded smooth domain E subject to the boundary condition $u = f$ on ∂E where f is a continuous function. We compare these solutions with a solution of the same boundary value problem for a linear equation $Lu = 0$. For the linear equation, we have

$$u(x) = \Pi_x f(\xi_{\tau_E})$$

where (ξ_t, Π_x) is an L-difusion. For the equation $Lu = \psi(u)$ an analogous formula can be written in terms of (L, ψ)-superdiffusion:

$$u(x) = -\log P_x e^{-\langle f, X_E \rangle}.$$

An expression in terms of $\mathbb{N}$-measures has the form

$$u(x) = \mathbb{N}_x(1 - e^{-\langle f, X_E \rangle}).$$

Because the absence of logarithm, this expression is closer than the previous one to the formula in the linear case. The dependence on x is more transparent and this opens new avenues for investigating the equation $Lu = \psi(u)$. To a great extent, Mselati's success in investigating the equation $\Delta u = u^2$ was achieved by following these avenues. Introducing $\mathbb{N}$-measures into the superdiffusion theory is a necessary step for extending his results to more general equations. In contrast to probability measures P_x, measures $\mathbb{N}_x$ are infinite (but they are σ-finite).

In this chapter we use shorter notation $\mathcal{M}$, $\mathcal{U}, \dots$ instead of notation $\mathcal{M}(E), \mathcal{U}(E), \dots$. No confusion should arise because we deal here with a fixed set E. We construct random measures $\mathbb{N}_x$ with the same auxiliary space $(\Omega, \mathcal{F})$ as the measures P_μ. We show that, for every $u \in \mathcal{U}^-$, the value Z_u can be chosen to satisfy **3**.(3.1) not only for P_μ but also for all $\mathbb{N}_x, x \in E$. Similarly, the range $\mathcal{R}_E$ can be chosen to be an envelope not only of $(\mathbb{S}_O, P_\mu)$ but also of $(\mathbb{S}_O, \mathbb{N}_x)$. We also give an expression for various elements of $\mathcal{U}$ in terms of measures $\mathbb{N}_x$.

1. Main result

1.1. We denote by $\mathbb{O}_x$ the class of open subsets of E which contain x and by $\mathcal{Z}_x$ the class of functions **3**.(2.1) with $D_i \in \mathbb{O}_x$. Put $Y \in \mathcal{Y}_x$ if $Y = e^{-Z}$ with $Z \in \mathcal{Z}_x$. In Theorem 1.1 and in section 2 we assume that $(E, \mathcal{B})$ is a *topological Luzin space*. [1]

The following result will be proved in Section 2.

THEOREM 1.1. *Suppose that $X = (X_D, P_\mu)$ is a canonical BEM system in $(E, \mathcal{B})$. For every $x \in E$, there exists a unique random measure $\mathbb{N}_x$ on the σ-algebra $\mathcal{F}^x$ generated by $X_O, O \in \mathbb{O}_x$ such that:*

1.1.A. For every $Y \in \mathcal{Y}_x$,

$$\mathbb{N}_x(1 - Y) = -\log P_x Y. \tag{1.1}$$

1.1.B.

$$\mathbb{N}_x(C) = 0$$

if $C \in \mathcal{F}^x$ is contained in the intersection of the sets $\{X_O = 0\}$ over all $O \in \mathbb{O}_x$.

Here we prove an immediate implication of this theorem.

COROLLARY 1.1. *For every $Z \in \mathcal{Z}_x$,*

$$\mathbb{N}_x\{Z \neq 0\} = -\log P_x\{Z = 0\}. \tag{1.2}$$

If $P_x\{Z = 0\} > 0$, then

$$\mathbb{N}_x\{Z \neq 0\} < \infty. \tag{1.3}$$

Equation (1.2) follows from (1.1) because $\lambda Z \in \mathcal{Z}_x$ for every $\lambda > 0$ and $1 - e^{-\lambda Z} \uparrow 1_{Z \neq 0}$ as $\lambda \to \infty$. Formula (1.3) follows from (1.2).

After we construct measures $\mathbb{N}_x$ in Section 2, we discuss their applications.

2. Construction of measures $\mathbb{N}_x$

2.1. Infinitely divisible random measures. Suppose that $(E, \mathcal{B})$ is a measurable space and let $X = (X(\omega), P)$ be a random measure with values in the space $\mathcal{M}$ of all finite measures on E. X is called *infinitely divisible* if, for every k, there exist independent identically distributed random measures $(X_1, P^{(k)}), \dots, (X_k, P^{(k)})$ such that the probability distribution of $X_1 + \dots + X_k$ under $P^{(k)}$ is the same as the probability distribution of X under P. This is equivalent to the condition

$$Pe^{-\langle f, X \rangle} = [P^{(k)} e^{-\langle f, X \rangle}]^k \quad \text{for every } f \in b\mathcal{B}. \tag{2.1}$$

[1] That is it is homeomorphic to a Borel subset $\tilde{E}$ of a compact metric space.

Denote by $\mathcal{B}_{\mathcal{M}}$ the σ-algebra in $\mathcal{M}$ generated by the sets $\{\nu : \nu(B) < c\}$ where $B \in \mathcal{B}, c \in \mathbb{R}$. It is clear that (2.1) is satisfied if, for all $f \in b\mathcal{B}$,

$$Pe^{-\langle f,X\rangle} = \exp\left[-\langle f, m\rangle - \mathcal{R}(1 - e^{-\langle f,\nu\rangle})\right] \tag{2.2}$$

where m is a measure on E and $\mathcal{R}$ is a measure on $(\mathcal{M}, \mathcal{B}_{\mathcal{M}})$. If $(E, \mathcal{B})$ is a *measurable Luzin space*, [2] then to every infinitely divisible random measure X there corresponds a pair $(m, \mathcal{R})$ subject to the condition (2.2) and this pair determines uniquely the probability distribution of X (see, e.g., [Ka77] or [Da93]). The right side in (2.2) does not depend on the value of $\mathcal{R}\{0\}$. If we put $\mathcal{R}\{0\} = 0$, then the pair $(m, \mathcal{R})$ is determined uniquely.

It follows from (2.2) that, for every constant $\lambda > 0$,

$$\lambda\langle 1, m\rangle + \mathcal{R}(1 - e^{-\lambda\langle 1,\nu\rangle}) = -\log Pe^{-\lambda\langle 1,X\rangle}.$$

The right side tends to $-\log P\{X = 0\}$ as $\lambda \to \infty$. Therefore if $P\{X = 0\} > 0$, then $m = 0$, $\mathcal{R}(\mathcal{M}) < \infty$ and (2.2) takes the form

$$Pe^{-\langle f,X\rangle} = \exp[-\mathcal{R}(1 - e^{-\langle f,\nu\rangle})]. \tag{2.3}$$

We call $\mathcal{R}$ the *canonical measure for X.*

2.2. Infinitely divisible random measures determined by a BEM system. Random measures (X_D, P_μ) which form a BEM system are infinitely divisible: the relation (2.1) holds with $P^{(k)} = P_{\mu/k}$. Moreover, to every family of open sets $I = \{D_1, \dots, D_n\}$ there corresponds an infinitely divisible measure (X_I, P_μ) on the union E_I of n replicas of E. Indeed, put

$$X_I = \{X_{D_1}, \dots, X_{D_n}\},\ f_I = \{f_1, \dots, f_n\},$$
$$\langle f_I, X_I\rangle = \sum_{i=1}^{n} \langle f_i, X_{D_i}\rangle \tag{2.4}$$

and use **3**.2.1.A and **3**.2.1.D to prove, by induction in n, that

$$P_\mu e^{-\langle f_I,X_I\rangle} = [P_{\mu/k} e^{-\langle f_I,X_I\rangle}]^k.$$

Therefore (X_I, P_μ) satisfies (2.1).

Note that, if $D \in \mathbb{O}_x$, then, by **3**.(2.7) and **2**.2.2.E,

$$P_x\{X_D = 0\} = \lim_{\lambda\to\infty} P_x e^{-\langle\lambda,X_D\rangle} = \lim_{\lambda\to\infty} e^{-V_D(\lambda)(x)} > 0.$$

It follows from **3**.2.1.G that, if $I = \{D_1, \dots, D_n\} \subset \mathbb{O}_x$, then $P_x\{X_I = 0\} > 0$.

Denote by $\mathcal{M}_I$ the space of all finite measures on E_I. There is a natural 1-1 correspondence between $\nu_I \in \mathcal{M}_I$ and collections $(\nu_1, \dots, \nu_n)$ where $\nu_i \in \mathcal{M}$. The product of n replicas of $\mathcal{B}_{\mathcal{M}}$ is a σ-algebra in $\mathcal{M}_I$. We denote it $\mathcal{B}_{\mathcal{M}_I}$. By applying formula (2.3), we get

$$P_x e^{-\langle f_I,X_I\rangle} = \exp[-\mathcal{R}_x^I(1 - e^{-\langle f_I,\nu_I\rangle})] \quad \text{for } I \subset \mathbb{O}_x \tag{2.5}$$

[2] That is if there exists a 1-1 mapping from E onto a topological Luzin space $\tilde{E}$ such that $B \in \mathcal{B}$ if and only if its image in $\tilde{E}$ is a Borel subset of $\tilde{E}$.

where $\mathcal{R}_x^I$ is a measure on $(\mathcal{M}_I, \mathcal{B}_{\mathcal{M}_I})$ not charging 0.

2.3. We use notation OI for the the family $\{O, D_1, \dots, D_n\}$ where $I = \{D_1, \dots, D_n\}$. We have:

2.3.A. If $OI \subset \mathbb{O}_x$, then for every f_I,

(2.6)
$$\mathcal{R}_x^{OI}\{\nu_O \neq 0, e^{-\langle f_I, \nu_I\rangle}\} = -\log P_x\{X_O = 0, e^{-\langle f_I, X_I\rangle}\} + \log P_x e^{-\langle f_I, X_I\rangle}.$$

Proof. Consider functions $f^\lambda = \{\lambda, f_1, \dots, f_n\}$ where $\lambda \geq 0$. By (2.5),

$$\begin{aligned}\mathcal{R}_x^{OI}\{-e^{-\langle f^\lambda, \nu_{OI}\rangle} + e^{-\langle f^0, \nu_{OI}\rangle}\}\\ &= \mathcal{R}_x^{OI}(1 - e^{-\lambda\langle 1, \nu_O\rangle - \langle f_I, \nu_I\rangle}) - \mathcal{R}_x^{OI}(1 - e^{-\langle f_I, \nu_I\rangle})\\ &= -\log P_x e^{-\lambda\langle 1, X_O\rangle - \langle f_I, X_I\rangle} + \log P_x e^{-\langle f_I, X_I\rangle}.\end{aligned}$$

Note that

$$\begin{aligned}-e^{-\langle f^\lambda, \nu_{OI}\rangle} + e^{-\langle f_I, \nu_I\rangle} &\to 1_{\{\nu_O \neq 0\}} e^{-\langle f_I, \nu_I\rangle},\\ e^{-\langle \lambda, X_O\rangle - \langle f_I, X_I\rangle} &\to 1_{\{X_O = 0\}} e^{-\langle f_I, X_I\rangle}\end{aligned}$$

as $\lambda \to \infty$ which implies (2.6). □

2.3.B. If $x \in O' \subset O$, then

$$P_x\{X_O = 0 | X_{O'} = 0\} = 1 \tag{2.7}$$

and

$$\mathcal{R}_x^{OO'}\{\nu_{O'} = 0, \nu_O \neq 0\} = 0. \tag{2.8}$$

Proof. By the Markov property **3**.2.1.D,

$$\begin{aligned}P_x\{X_{O'} = 0\} - P_x\{X_{O'} = X_O = 0\} &= P_x\{X_{O'} = 0, X_O \neq 0\}\\ &= P_x[X_{O'} = 0, P_{X_{O'}}\{X_O \neq 0\}] = 0\end{aligned}$$

which implies (2.7).

By 2.3.A,

$$\mathcal{R}_x^{OO'}\{\nu_O \neq 0, e^{-\langle \lambda, \nu_{O'}\rangle}\} = -\log P_x\{X_O = 0, e^{-\langle \lambda, X_{O'}\rangle}\} + \log P_x e^{-\langle \lambda, X_{O'}\rangle}.$$

By passing to the limit as $\lambda \to \infty$, we get

$$\mathcal{R}_x^{OO'}\{\nu_{O'} = 0, \nu_O \neq 0\} = -\log P_x\{X_{O'} = 0, X_O = 0\} + \log P_x\{X_{O'} = 0\}$$

and therefore (2.8) follows from (2.7). □

2.3.C. If $I \subset J \subset \mathbb{O}_x$, then

$$\mathcal{R}_x^{OI}\{\nu_O \neq 0, \nu_I \in B\} = \mathcal{R}_x^{OJ}\{\nu_O \neq 0, \nu_I \in B\}$$

for every $B \in \mathcal{B}_{\mathcal{M}_I}$.

PROOF. Suppose that $f_{J\setminus I} = 0$. Since $\langle f_I, X_I\rangle = \langle f_J, X_J\rangle$, we conclude from (2.6) that

$$\mathcal{R}_x^{OI}\{\nu_O \neq 0, e^{-\langle f_I,\nu_I\rangle}\} = \mathcal{R}_x^{OJ}\{\nu_O \neq 0, e^{-\langle f_J,\nu_J\rangle}\}. \tag{2.9}$$

By the *Multiplicative systems theorem* (see, e. g., [**D**], the Appendix A), this implies 2.3.C. □

2.4. Proof of Theorem 1.1. 1°. Note that, by (2.6), $\mathcal{R}_x^{OI}(\nu_O \neq 0) = -\log P_x\{X_O = 0\}$ does not depend on I. It is finite because $P_x\{X_O = 0\} > 0$. Consider a set $\Omega_O = \{X_O \neq 0\}$ and denote by $\mathcal{F}_O$ the σ-algebra in Ω_O generated by $X_D(\omega), D \in \mathbb{O}_x$. It follows from 2.3.C and Kolmogorov's theorem about measures on functional spaces that there exists a unique measure $\mathbb{N}_x^O$ on $(\Omega_O, \mathcal{F}_O)$ such that

$$\mathbb{N}_x^O e^{-\langle f_I,X_I\rangle} = \mathcal{R}_x^{OI}\{\nu_O \neq 0, e^{-\langle f_I,\nu_I\rangle}\} \tag{2.10}$$

for all I and all f_I.

By the Multiplicative systems theorem,

$$\mathbb{N}_x^O F(X_I) = \mathcal{R}_x^{OI}\{\nu_O \neq 0, F(\nu_I)\} \tag{2.11}$$

for every positive measurable F.

2°. Suppose that $x \in O' \subset O$. We claim that $\Omega_O \subset \Omega_{O'}$ $\mathbb{N}_x^O$-a.s. and that $\mathbb{N}_x^O = \mathbb{N}_x^{O'}$ on Ω_O. The first part holds because, by (2.11) and 2.3.B,

$$\mathbb{N}_x^O\{X_{O'} = 0\} = \mathcal{R}_x^{OO'}\{\nu_O \neq 0, \nu_{O'} = 0\} = 0.$$

The second part follows from the relation

$$\mathbb{N}_x^O\{X_O \neq 0, F(X_I)\} = \mathbb{N}_x^{O'}\{X_O \neq 0, F(X_I)\} \tag{2.12}$$

for all positive measurable F. To prove this relation we observe that, by (2.11),

$$\mathbb{N}_x^{O'}\{X_O \neq 0, F(X_I)\} = \mathcal{R}_x^{O'OI}\{\nu_{O'} \neq 0, \nu_O \neq 0, F(\nu_I)\}. \tag{2.13}$$

By (2.11) and 2.3.C

(2.14)

$$\mathbb{N}_x^O\{X_O \neq 0, F(X_I)\} = \mathcal{R}_x^{OI}\{\nu_O \neq 0, F(\nu_I)\} = \mathcal{R}_x^{OO'I}\{\nu_O \neq 0, F(\nu_I)\}.$$

By 2.3.C and 2.3.B,

$$\mathcal{R}_x^{OO'I}\{\nu_O \neq 0, \nu_{O'} = 0\} = \mathcal{R}_x^{OO'}\{\nu_O \neq 0, \nu_{O'} = 0\} = 0.$$

Therefore the right sides in (2.13) and (2.14) are equal.

3°. Note that, for every $O_1, O_2 \in \mathbb{O}_x$, $\mathbb{N}_x^{O_1} = \mathbb{N}_x^{O_2}$ on $\Omega_{O_1} \cap \Omega_{O_2}$ because, for $O' = O_1 \cap O_2$, $\mathbb{N}_x^{O_1} = \mathbb{N}_x^{O'}$ on Ω_{O_1} and $\mathbb{N}_x^{O_2} = \mathbb{N}_x^{O'}$ on Ω_{O_2}. Let Ω^* be the union of Ω_O over all $O \in \mathbb{O}_x$. There exists a measure $\mathbb{N}_x$ on Ω^* such that

$$\mathbb{N}_x = \mathbb{N}_x^O \quad \text{on } \Omega_O \text{ for every } O \in \mathbb{O}_x. \tag{2.15}$$

By setting $\mathbb{N}_x(C) = 0$ for every $C \subset \Omega \setminus \Omega^*$ which belongs to $\mathcal{F}^x$ we satisfy condition 1.1.B of our theorem.

4°. It remains to prove that $\mathbb{N}_x$ satisfies condition 1.1.A. We need to check that

$$\mathbb{N}_x\{1 - e^{-\langle f_I, X_I\rangle}\} = -\log P_x e^{-\langle f_I, X_I\rangle} \tag{2.16}$$

for every $I = \{D_1, \dots, D_n\}$ such that $D_i \in \mathbb{O}_x$ and for every f_I. The intersection O of D_i belongs to $\mathbb{O}_x$. Since, for all i, $\{X_O = 0\} \subset \{X_{D_i} = 0\}$ $\mathbb{N}_x$-a.s., we have

$$\{X_O = 0\} \subset \{e^{-\langle f_I, X_I\rangle} = 1\} \quad \mathbb{N}_x - \text{a.s.} \tag{2.17}$$

and

$$\mathbb{N}_x\{1 - e^{-\langle f_I, X_I\rangle}\} = \mathbb{N}_x\{X_O \neq 0, 1 - e^{-\langle f_I, X_I\rangle}\} = \mathbb{N}_x^O\{1 - e^{-\langle f_I, X_I\rangle}\}.$$

By (2.11), the right side is equal to $\mathcal{R}_x^{OI}\{\nu_O \neq 0, 1 - e^{-\langle f_I, \nu_I\rangle}\}$. This is equal to $-\log P_x e^{-\langle f_I, X_I\rangle}$ by (2.6)and (2.17).

5°. If two measures $\mathbb{N}_x$ and $\tilde{\mathbb{N}}_x$ satisfy the condition 1.1.A, then

$$\mathbb{N}_x\{X_O \neq 0, 1 - Y\} = \tilde{\mathbb{N}}_x\{X_O \neq 0, 1 - Y\} \tag{2.18}$$

for all $O \in \mathbb{O}_x$ and all $Y \in \mathcal{Y}_x$. (This can be proved by a passage to the limit similar to one used in the proof of Corollary 1.1.) The family $\{1 - Y, Y \in \mathcal{Y}_x\}$ is closed under multiplication. By the Multiplicative systems theorem, (2.18) implies that $\mathbb{N}_x\{X_O \neq 0, C\} = \tilde{\mathbb{N}}_x\{X_O \neq 0, C\}$ for every $C \in \mathcal{F}^x$ contained in Ω^*. By (1.1.B), $\mathbb{N}_x(C) = \tilde{\mathbb{N}}_x(C) = 0$ for $C \in \mathcal{F}^x$ contained in $\Omega \setminus \Omega^*$. Thus $\mathbb{N}_x = \tilde{\mathbb{N}}_x$ on $\mathcal{F}^x$. □

3. Applications

3.1. Now we consider an (L, ψ)-superdiffusion (X_D, P_μ) in a domain $E \subset \mathbb{R}^d$. All these superdiffusions satisfy the condition

$$0 < P_x\{X_D = 0\} < 1 \quad \text{for every } D \subset E \text{ and every } x \in D. \tag{3.1}$$

By **2**.2.2.C, if $u \in \mathcal{U}$ then $V_D(u) = u$ for every $D \Subset E$.

3.2. Stochastic boundary value.

Theorem 3.1. *Let $X = (X_D, P_\mu)$ be an (L, ψ)-superdiffusion. For every $u \in \mathcal{U}^-$, there exists a function $Z_u(\omega)$ such that*

(3.2)

$$\lim \langle u, X_{D_n}\rangle = Z_u \quad P_\mu\text{-a.s. for all } \mu \in \mathcal{M}(E) \text{ and } \quad \mathbb{N}_x\text{-a.s.for all } x \in E$$

for every sequence D_n exhausting E. [3]

From now on we use the name a *stochastic boundary value* of u and the notation SBV(u) for Z_u which satisfies (3.2).

To prove Theorem 3.1 we use two lemmas.

[3] $\langle u, X_{D_n}\rangle \in \mathcal{F}^x$ for all sufficiently big n.

LEMMA 3.1. *For every* $Z, \tilde{Z} \in \mathcal{Z}_x$,

$$\mathbb{N}_x\{\tilde{Z} = 0, Z \neq 0\} = -\log P_x\{Z = 0 | \tilde{Z} = 0\} \tag{3.3}$$

If $x \in O' \subset O$, *then*

$$\{X_O \neq 0\} \subset \{X_{O'} \neq 0\} \quad \mathbb{N}_x\text{-}a.s. \tag{3.4}$$

PROOF. By (1.2),

$$\mathbb{N}_x\{\tilde{Z} \neq 0\} = -\log P_x\{\tilde{Z} = 0\}$$

and

$$\mathbb{N}_x\{\tilde{Z} + Z \neq 0\} = -\log P_x\{\tilde{Z} + Z = 0\} = -\log P_x\{\tilde{Z} = 0, Z = 0\}.$$

Therefore

$$\begin{aligned}\mathbb{N}_x\{\tilde{Z} = 0, Z \neq 0\} &= \mathbb{N}_x\{\tilde{Z} + Z \neq 0\} - \mathbb{N}_x\{\tilde{Z} \neq 0\} \\ &= -\log P_x\{\tilde{Z} = 0, Z = 0\} + \log P_x\{\tilde{Z} = 0\}\end{aligned}$$

which implies (3.3). Formula (3.4) follows from (3.3) and (2.7). □

Denote $\mathcal{F}^x_{\subset D}$ the σ-algebra generated by $X_{D'}$ such that $x \in D' \subset D$.

LEMMA 3.2. *Put* $Y_O = e^{-\langle u, X_O \rangle}$. *If* $u \in \mathcal{U}^-$ *and* $x \in O' \subset O$, *then, for every* $V \in \mathcal{F}^x_{\subset O'}$,

$$\mathbb{N}_x\{X_{O'} \neq 0, V(1 - Y_O)\} \leq \mathbb{N}_x\{X_{O'} \neq 0, V(1 - Y_{O'})\}. \tag{3.5}$$

PROOF. Note that

$$\mathbb{N}_x\{X_{O'} \neq 0, V(1 - Y_O)\} = \mathbb{N}_x V(1 - Y_O). \tag{3.6}$$

Indeed,

$$1_{\{X_{O'} \neq 0\}}(1 - Y_O) = 1 - Y_O$$

on $\{X_{O'} \neq 0\}$. By (3.4), this equation holds $\mathbb{N}_x$-a.s. on $\{X_O \neq 0\}$. It holds also on $\{X_O = 0\}$ because there both sides are equal to 0.

To prove our lemma, it is sufficient to show that (3.5) holds for $V = e^{-\langle f_I, X_I \rangle}$ with $I = \{D_1, \dots, D_n\}$ where $x \in D_i \subset O'$. By (3.6) and (1.1),

$$\begin{aligned}&\mathbb{N}_x\{X_{O'} \neq 0, V(Y_O - Y_{O'})\} \\ &= \mathbb{N}_x\{X_{O'} \neq 0, V(1 - Y_{O'})\} - \mathbb{N}_x\{X_{O'} \neq 0, V(1 - Y_O)\} \\ &= \mathbb{N}_x\{V(1 - Y_{O'})\} - \mathbb{N}_x\{V(1 - Y_O)\} = -\mathbb{N}_x(1 - VY_O) + \mathbb{N}_x(1 - VY_{O'}) \\ &= -\log P_x VY_{O'} + \log P_x VY_O.\end{aligned} \tag{3.7}$$

If $u \in \mathcal{U}^-$, then $P_\mu Y_O = e^{-\langle V_O(u), \mu \rangle} \geq e^{-\langle u, \mu \rangle}$ and, by the Markov property **3**.2.1.D,

$$P_x VY_O = P_x(VP_{X_{O'}} Y_O) \geq P_x VY_{O'}.$$

Therefore the right side in (3.7) is bigger than or equal to 0 which implies (3.5). □

3.3. Proof of Theorem 3.1. As we know (see Theorem **3**.3.2), the limit **3**.(3.1) exists P_μ-a.s. and is independent of a sequence D_n. Let us prove that this limit exists also $\mathbb{N}_x$-a.s.

Put $\Omega_m = \{X_{D_m} \neq 0\}$, $Y_n = e^{-\langle u, X_{D_n}\rangle}$. If m is sufficiently large, then $D_m \in \mathbb{O}_x$. For every such m and for all $n \geq m$, denote by $\mathcal{F}_n^m$ the σ-algebra in Ω_m generated by X_U where $x \in U \subset D_n$. It follows from (1.2) and (3.1) that

$$0 < \mathbb{N}_x(\Omega_m) < \infty.$$

The formula

$$Q_x^m(C) = \frac{\mathbb{N}_x(C)}{\mathbb{N}_x(\Omega_m)}$$

defines a probability measure on Ω_m. By Lemma 3.2 applied to $O' = D_n$ and $O = D_{n+1}$,

$$\mathbb{N}_x\{\Omega_n, V(1 - Y_{n+1})\} \leq \mathbb{N}_x\{\Omega_n, V(1 - Y_n)\} \quad \text{for } V \in \mathcal{F}_{\subset D_n}$$

and therefore

$$Q_x^m\{V(1 - Y_{n+1})\} \leq Q_x^m\{V(1 - Y_n)\} \quad \text{for } n \geq m \quad \text{and } V \in \mathcal{F}_n^m.$$

Hence, $1 - Y_n, n \geq m$ is a supermartingale relative to $\mathcal{F}_n^m$ and Q_x^m. We conclude that, Q_x^m-a.s., there exists $\lim(1 - Y_n)$ and therefore there exists also the limit **3**.(3.1). □

3.4.

THEOREM 3.2. *If $Z = Z^0 + Z_u$ where $Z^0 \in \mathcal{Z}_x, u \in \mathcal{U}^-$, then*

$$\mathbb{N}_x(1 - e^{-Z}) = -\log P_x e^{-Z}. \tag{3.8}$$

First we prove a lemma. For every $U \in \mathbb{O}_x$, denote by $\mathcal{Z}_U$ the class of functions **3**.(2.1) with $D_i \supset U$ and put $Y \in \mathcal{Y}_U$ if $Y = e^{-Z}$ with $Z \in \mathcal{Z}_U$.

LEMMA 3.3. *Suppose that U is a neighborhood of x. If $Y_n \in \mathcal{Y}_U$ converge P_x-a.s. to Y and if $P_x\{Y > 0\} > 0$, then*

$$\mathbb{N}_x(1 - Y) = -\log P_x Y. \tag{3.9}$$

PROOF. By the Markov property **3**.2.1.D, $P_x\{X_U = 0, X_D \neq 0\} = 0$ for every $D \supset U$ and therefore every $Y \in \mathcal{Y}_U$ is equal to 1 P_x-a.s. on $C = \{X_U = 0\}$.

Denote by Q the restriction of $\mathbb{N}_x$ to $\{X_U \neq 0\}$. By (2.6), (2.10) and (2.15), if $Y \in \mathcal{Y}_U$, then

$$QY = -\log P_x\{C, Y\} + \log P_x Y = -\log P_x(C) + \log P_x Y. \tag{3.10}$$

Since $Y_m^2, Y_n^2, Y_m Y_n$ belong to $\mathcal{Y}_U$, we have

$$Q(Y_m - Y_n)^2 = QY_m^2 + QY_n^2 - 2QY_mY_n = \log P_x Y_m^2 + \log P_x Y_n^2 - 2\log P_x Y_m Y_n.$$

By the dominated convergence theorem, the right side tends to 0 as $m, n \to \infty$. A subsequence Y_{n_k} converges P_x-a.s. and Q-a.s. to Y. Since Q is a finite measure and $0 \leq Y_n \leq 1$,

$$QY_{n_k} \to QY.$$

Formula (3.10) holds for Y_n. By passing to the limit, we conclude that it holds for Y. Therefore $\mathbb{N}_x\{Y, X_U \neq 0\} = -\log P_x(C) + \log P_x Y$. By (1.2), this implies (3.9). □

Proof of Theorem 3.2. If D_n exhaust E, then, P_x-a.s., $Y = e^{-Z} = \lim Y_n$ where $Y_n = e^{-Z^0 - \langle u, X_{D_n}\rangle} \in \mathcal{Y}_x$. For some $U \in \mathbb{O}_x$, all Y_n belong to $\mathcal{Y}_U$. It remains to check that $P_x\{Y > 0\} > 0$. Note that $Z^0 < \infty$ P_x-a.s. and

$$P_x e^{-\langle u, X_{D_n}\rangle} = e^{-V_{D_n}(u)(x)} \geq e^{-u(x)}.$$

Therefore $P_x e^{-Z_u} > 0$ and $P_x\{Z_u < \infty\} > 0$. □

REMARK 3.1. It follows from Theorem 3.2 that, for every $\nu \in \mathcal{M}(E)$,

$$\mathbb{N}_x Z_\nu = P_x Z_\nu.$$

Indeed, for every $\lambda > 0$, $u = \lambda h_\nu \in \mathcal{U}^-$ and therefore, by (3.8), $\mathbb{N}_x(1 - e^{-\lambda Z_\nu}) = -\log P_x e^{-\lambda Z_\nu}$. Since $P_x Z_\nu < \infty$ by **3**.3.6.A, we can differentiate under the integral signs.

3.5. Range.

THEOREM 3.3. *For every $x \in E$, a closed set $\mathcal{R}_E$ can be chosen to be, at the same time, an envelope of the family $(\mathbb{S}_O, P_x), O \subset E$ and an envelope of the family $(\mathbb{S}_O, \mathbb{N}_x), O \in \mathbb{O}_x$. For every Borel subset Γ of ∂E,*

$$\mathbb{N}_x\{\mathcal{R}_E \cap \Gamma \neq \emptyset\} = -\log P_x\{\mathcal{R}_E \cap \Gamma = \emptyset\}. \tag{3.11}$$

The following lemma is needed to prove Theorem 3.3.

LEMMA 3.4. *Suppose that U is a relatively open subset of ∂E, O is an open subset of E, O_k exhaust O and*

$$A_U = \{X_{O_k}(U) = 0 \quad \textit{for all } k, X_O(U) \neq 0\}. \tag{3.12}$$

Then $P_\mu(A_U) = 0$ for all $\mu \in \mathcal{M}(E)$ and $\mathbb{N}_x(A_U) = 0$ for all $x \in O$.

PROOF. By [**D**], Lemma 4.5.1, $P_\mu(A_U) = 0$ for $\mu \in \mathcal{M}(E)$. If $x \in O$, then $x \in O_m$ for some m. Since the sequence $O_m, O_{m+1}, \dots$ exhaust O, we can assume that $x \in O_1$. Put $Z = X_O(U), \tilde{Z}_n = \sum_1^n X_{O_k}(U)$ and note that $A_U = \{\tilde{Z}_\infty = 0, Z \neq 0\}$ and $P_x\{\tilde{Z}_\infty = 0\} \geq P_x\{X_{O_1} = 0\} > 0$. By Lemma 3.1 applied to Z and $\tilde{Z}_n$,

$$\mathbb{N}_x\{A_U\} \leq \mathbb{N}_x\{\tilde{Z}_n = 0, Z \neq 0\} = -\log P_x\{Z = 0 | \tilde{Z}_n = 0\}.$$

As $n \to \infty$, the right side tends to

$$-\log\{1 - P_x(A_U)/P_x[\tilde{Z}_\infty = 0]\} = 0.$$

Hence $\mathbb{N}_x A_U = 0$. □

3.6. Proof of Theorem 3.3. 1°. We prove the first part of the theorem by using the construction described in Section **3**.2.4. It follows from Lemma 3.4 that the support $\mathcal{R}_E$ of the measure M defined by **3**.(2.18) is a minimal closed set which contains, P_μ-a.s. for $\mu \in \mathcal{M}(E)$ and $\mathbb{N}_x$-a.s., the support of every measure $X_D, D \in \mathbb{O}_x$. The proof is identical to the proof of Theorem 5.1 in [**D**], p. 62 or Theorem 5.1 in [Dy98], p. 174.

2°. First, we prove formula (3.11) for relatively open subsets of ∂E. For every such a subset U, we put

$$\begin{gathered} Z_k = X_{O_k}(U), \quad \tilde{Z}_n = \sum_1^n Z_k, \\ A_1 = \{Z_1 \neq 0\}, \quad A_n = \{\tilde{Z}_{n-1} = 0, Z_n \neq 0\} \quad \text{for } n > 1. \end{gathered} \tag{3.13}$$

Note that

$$\begin{gathered} \{\mathcal{R}_E \cap U = \emptyset\} = \{M(U) = 0\} = \{Z_n = 0 \quad \text{for all } n\}, \\ \{\mathcal{R}_E \cap U \neq \emptyset\} = \bigcup A_n \end{gathered} \tag{3.14}$$

and $P_x\{\tilde{Z}_n = 0\} > 0$ for all n. By Lemma 3.1 applied to $Z = Z_n$ and $\tilde{Z} = \tilde{Z}_{n-1}$, we have

$$\mathbb{N}_x(A_n) = -\log P_x\{Z_n = 0 | \tilde{Z}_{n-1} = 0\}$$

and therefore, by (3.14),

$$\begin{aligned} \mathbb{N}_x\{\mathcal{R}_E \cap U \neq \emptyset\} &= -\log \prod_1^\infty P_x\{Z_n = 0 | \tilde{Z}_{n-1} = 0\} \\ &= -\log P_x\{Z_n = 0 \quad \text{for all } n\} = -\log P_x\{\mathcal{R}_E \cap U = \emptyset\}. \end{aligned} \tag{3.15}$$

Thus formula (3.11) holds for open sets.

Now suppose that K is a closed subset of ∂E and let $U_n = \{x \in \partial E : d(x, K) < 1/n\}$. By applying (3.15) to U_n and by passing to the limit, we prove that (3.11) is satisfied for K.

To extend (3.11) to all Borel sets $\Gamma \subset \partial E$, we consider Choquet capacities [4]

$$\text{Cap}_1(\Gamma) = P_x\{\mathcal{R}_E \cap \Gamma \neq \emptyset\}$$

and

$$\text{Cap}_2(\Gamma) = \mathbb{N}_x\{\mathcal{R}_E \cap \Gamma \neq \emptyset\}.$$

[Note that $\text{Cap}_2(\Gamma) \leq \text{Cap}_2(\partial E) = -\log P_x\{\mathcal{R}_E \cap \partial E = \emptyset\} < \infty$.] There exists a sequence of compact sets K_n such that $\text{Cap}_1(K_n) \to \text{Cap}_1(\Gamma)$ and $\text{Cap}_2(K_n) \to \text{Cap}_2(\Gamma)$. We have

$$\text{Cap}_2(K_n) = -\log[1 - \text{Cap}_1(K_n)].$$

By passing to the limit we prove that (3.11) holds for Γ. □

[4] See Section **2**.4.

REMARK. A new probabilistic formula

$$w_\Gamma(x) = \mathbb{N}_x\{\mathcal{R}_E \cap \Gamma \neq \emptyset\}. \tag{3.16}$$

for functions defined by **1**.(1.5)–**1**.(1.6) follows from (3.11) and **3**.(3.9).

3.7. Probabilistic expression of a solution through its trace.

THEOREM 3.4. *If $Z = SBV(u)$ for $u \in \mathcal{U}^-$, then, for every Borel set $\Gamma \subset \partial E$,*

$$-\log P_x\{\mathcal{R}_E \cap \Gamma = \emptyset, e^{-Z}\} = \mathbb{N}_x\{\mathcal{R}_E \cap \Gamma \neq \emptyset\} + \mathbb{N}_x\{\mathcal{R}_E \cap \Gamma = \emptyset, 1 - e^{-Z}\}. \tag{3.17}$$

Formula (3.17) with $Z = Z_\nu, \nu \in \mathcal{N}_0^E$ provides a probabilistic expression for the solution $w_\Gamma \oplus u_\nu$. In particular,

$$-\log P_x e^{-Z_\nu} = \mathbb{N}_x\{1 - e^{-Z_\nu}\} = u_\nu(x) \tag{3.18}$$

and

$$-\log P_x\{\mathcal{R}_E \cap \Gamma = \emptyset\} = \mathbb{N}_x\{\mathcal{R}_E \cap \Gamma \neq \emptyset\} = w_\Gamma(x). \tag{3.19}$$

3.8. In preparation for proving Theorem 3.4 we establish the following result.

LEMMA 3.5. *If $Z = \mathrm{SBV}(u), u \in \mathcal{U}^-$, then for every $Z', Z'' \in \mathcal{Z}_x$,*

$$\mathbb{N}_x\{Z' = 0, 1 - e^{-Z}\} = -\log P_x\{e^{-Z} | Z' = 0\} \tag{3.20}$$

and

$$\begin{aligned} &\mathbb{N}_x\{Z' = 0, Z'' \neq 0, e^{-Z}\} \\ &\qquad = -\log P_x\{e^{-Z}|Z' = 0\} + \log P_x\{e^{-Z}|Z' = Z'' = 0\}. \end{aligned} \tag{3.21}$$

PROOF. By Theorem 3.2, for every $\lambda > 0$,

$$-\log P_x e^{-\lambda Z' - Z} = \mathbb{N}_x(1 - e^{-\lambda Z' - Z}).$$

By taking $\lambda \to \infty$, we get

$$-\log P_x\{Z' = 0, e^{-Z}\} = \mathbb{N}_x(1 - 1_{Z'=0} e^{-Z}).$$

By (1.2), this implies (3.20). Note that

$$\{Z' = 0, Z'' \neq 0\} = \{Z' = 0\} \setminus \{Z' + Z'' = 0\}.$$

Therefore

$$\mathbb{N}_x\{Z' = 0, Z'' \neq 0, 1 - e^{-Z}\} = \mathbb{N}_x\{Z' = 0, 1 - e^{-Z}\} - \mathbb{N}_x\{Z' + Z'' = 0, 1 - e^{-Z}\}$$

and we get (3.21) by applying (3.20). □

3.9. Proof of Theorem 3.4. We use notation (3.13). Put

$$I_n = -\log P_x\{e^{-Z}|\tilde{Z}_n = 0\}.$$

By (3.14),

$$(3.22)\quad I_\infty = \lim_{n\to\infty} I_n = -\log P_x\{e^{-Z}|\mathcal{R}_E \cap U = \emptyset\}$$
$$= -\log P_x\{\mathcal{R}_E \cap U = \emptyset, e^{-Z}\} + \log P_x\{\mathcal{R}_E \cap U = \emptyset\}.$$

By (3.22) and (3.11),

$$(3.23)\qquad -\log P_x\{\mathcal{R}_E \cap U = \emptyset, e^{-Z}\} = I_\infty + \mathbb{N}_x\{\mathcal{R}_E \cap U \neq \emptyset\}.$$

By (3.14),

$$(3.24)\qquad \mathbb{N}_x\{\mathcal{R}_E \cap U \neq \emptyset, 1 - e^{-Z}\} = \sum_1^\infty \mathbb{N}_x\{A_n, 1 - e^{-Z}\}.$$

It follows from (3.20) and (3.21) that

$$\mathbb{N}_x\{A_1, 1 - e^{-Z}\} = -\log P_x e^{-Z} - I_1$$

and

$$\mathbb{N}_x\{A_n, 1 - e^{-Z}\} = I_{n-1} - I_n \quad \text{for } n > 1.$$

Therefore

$$(3.25)\ \mathbb{N}_x\{\mathcal{R}_E \cap U \neq \emptyset, 1 - e^{-Z}\} = \sum_1^\infty \mathbb{N}_x\{A_n, 1 - e^{-Z}\} = -\log P_x e^{-Z} - I_\infty$$

and, by (3.8),

$$(3.26)\quad I_\infty = -\log P_x e^{-Z} - \mathbb{N}_x\{\mathcal{R}_E \cap U \neq \emptyset, 1 - e^{-Z}\}$$
$$= \mathbb{N}_x(1 - e^{-Z}) - \mathbb{N}_x\{\mathcal{R}_E \cap U \neq \emptyset, 1 - e^{-Z}\} = \mathbb{N}_x\{\mathcal{R}_E \cap U = \emptyset, 1 - e^{-Z}\}.$$

It follows from (3.23) and (3.26) that (3.17) is true for open sets Γ. An extension to all Borel sets can be done in the same way as in the proof of Theorem 3.3.

To prove the second part of the theorem, it is sufficient to show that

$$(3.27)\qquad w_\Gamma \oplus u_\nu = -\log P_x\{\mathcal{R}_E \cap \Gamma = \emptyset, e^{-Z_\nu}\}.$$

Let $u = w_\Gamma \oplus u_\nu$. By **3**.3.3.B, $\mathrm{SBV}(u) = Z_\Gamma + Z_\nu$ where $Z_\Gamma = \mathrm{SBV}(w_\Gamma)$. By **3**.(3.4), $u(x) = -\log P_x e^{-Z_\Gamma - Z_\nu}$, and (3.27) follows from **3**.(3.5.A). □

3.10. It follows from (3.17) and (3.11) that

$$(3.28)\qquad \mathbb{N}_x\{\mathcal{R}_E \cap \Gamma = \emptyset, 1 - e^{-Z}\} = -\log P_x\{e^{-Z} \mid \mathcal{R}_E \cap \Gamma = \emptyset\}.$$

[By **3**.(3.9), $P_x\{\mathcal{R}_E \cap \Gamma = \emptyset\} = e^{-w_\Gamma(x)} > 0$.]

By applying (3.28) to λZ and by passing to the limit as $\lambda \to +\infty$, we get

$$(3.29)\qquad \mathbb{N}_x\{\mathcal{R}_E \cap \Gamma = \emptyset, Z \neq 0\} = -\log P_x\{Z = 0 \mid \mathcal{R}_E \cap \Gamma = \emptyset\}.$$

If $\nu \in \mathcal{N}_1^E$ is concentrated on Γ, then $\{\mathcal{R}_E \cap \Gamma = \emptyset\} \subset \{Z_\nu = 0\}$ P_x-a.s. and therefore

$$\mathbb{N}_x\{\mathcal{R}_E \cap \Gamma = \emptyset, Z_\nu \neq 0\} = 0. \tag{3.30}$$

It follows from (3.29) and (3.11) that

$$-\log P_x\{\mathcal{R}_E \cap \Gamma = \emptyset, Z = 0\} = \mathbb{N}_x\{\mathcal{R}_E \cap \Gamma \neq \emptyset\} + \mathbb{N}_x\{\mathcal{R}_E \cap \Gamma = \emptyset, Z \neq 0\}. \tag{3.31}$$

We conclude from this relation and **3**.(3.14) that

$$u_{\infty \cdot \nu} = -\log P_x\{Z_\nu = 0\} = \mathcal{N}_x\{Z_\nu \neq 0\}. \tag{3.32}$$

4. Notes

The results presented in this chapter can be found in [DK04].

A systematic presentation of Le Gall's theory of the Brownian snake and its applications to a semilinear equation $\Delta u = u^2$ is contained in his book [Le99]. It starts with a direct construction of the snake. A related (L, ψ)-superdiffusion with quadratic branching $\psi(u) = u^2$ is defined by using the local times of the snake. A striking example of the power of this approach is Wiener's test for the Brownian snake (first, published in [DL97]) that yields a complete characterization of the domains in which there exists a solution of the problem

$$\begin{aligned} &\Delta u = u^2 \quad \text{in } E, \\ &u = \infty \quad \text{on } \partial E. \end{aligned}$$

Only partial results in this direction were obtained before by analysts. [5]

A more general path-valued process – the Lévy snake was studied in a series of papers of Le Gall and Le Jan. Their applications to constructing (ξ, ψ)-superprocesses for a rather wide class of ψ are discussed in Chapter 4 of the monograph [DuL02].

We refer to the bibliography on the Brownian snake and the Lévy snake in [Le99] and [DuL02].

[5] Later Labutin [Lab03] proved a similar result for all equations $\Delta u = u^\alpha$ with $\alpha > 1$ by analytical methods.

CHAPTER 5

Moments and absolute continuity properties of superdiffusions

In this chapter we consider (L, ψ)-superdiffusions in an arbitrary domain E, with ψ defined by **3**.(2.8) subject to the condition **3**.(2.9).

The central result (which will be used in Chapter **9**) is that, if A belongs to the germ σ-algebra $\mathcal{F}_\partial$ (defined in Section 3.4 of Chapter **3**), then either $P_\mu(A) = 0$ for all $\mu \in \mathcal{M}_c(E)$ or $P_\mu(A) > 0$ for all $\mu \in \mathcal{M}_c(E)$. The proof is based on the computation of the integrals

$$\int e^{-\langle f_0, X_D\rangle} \langle f_1, X_D\rangle \dots \langle f_n, X_D\rangle \tag{0.1}$$

with respect to measures $\mathbb{N}_x$ and P_μ and on a Poisson representation of infinitely divisible measures.

As an intermediate step we consider the surface area γ on the boundary of a smooth domain D and we prove that the measures

$$n_D^x(B) = \mathbb{N}_x \int_B X_D(dy_1) \dots X_D(dy_n), \quad x \in D \tag{0.2}$$

and

$$p_D^\mu(B) = P_\mu \int_B X_D(dy_1) \dots X_D(dy_n), \quad \mu \in \mathcal{M}_c(D) \tag{0.3}$$

vanish on the same class of sets B as the product measure $\gamma^n(dy_1, \dots, dy_n) = \gamma(dy_1) \dots \gamma(dy_n)$.

1. Recursive moment formulae

Let $D \Subset E$ and let $f_0, f_1, \dots \in \mathcal{B}(\bar{D})$. Put

$$\ell = \psi'[V_D(f_0)]. \tag{1.1}$$

We express the integrals (0.1) through the operators $G_D^\ell f(x)$ and $K_D^\ell f(x)$ defined by **3**.(1.5) and a sequence

$$\begin{aligned} &q_1(x) = 1,\ q_2(x) = 2b + \int_0^\infty t^2 e^{-t\ell(x)} N(x, dt), \\ &q_r(x) = \int_0^\infty t^r e^{-t\ell(x)} N(x, dt) \quad \text{for } r > 2 \end{aligned} \tag{1.2}$$

which we call a *q-sequence*. By **3**.(2.11), the function $\psi(x,u)$ is infinitely differentiable with respect to u and

$$q_r(x) = (-1)^r \psi_r(x, \ell(x)) \quad \text{for } r \geq 2. \tag{1.3}$$

The functions q_r are strictly positive and bounded.

1.1. Results. We consider nonempty finite subsets $C = \{i_1, \dots, i_n\}$ of the set $\{1, 2, \dots\}$ and we put $|C| = n$. We denote by $\mathfrak{P}_r(C)$ the set of all partitions of C into r disjoint nonempty subsets $C_1, \dots, C_r$. We do not distinguish partitions obtained from each other by permutations of $C_1, \dots, C_r$ and by permutations of elements inside each C_i. For instance, for $C = \{1, 2, 3\}$, the set $\mathfrak{P}_2(C)$ consists of three elements $\{1, 2\} \cup \{3\}$, $\{1, 3\} \cup \{2\}$ and $\{2, 3\} \cup \{1\}$. We denote by $\mathfrak{P}(C)$ the union of $\mathfrak{P}_r(C)$ over $r = 1, 2, \dots, |C|$.

For any functions $\varphi_i \in \mathcal{B}(\bar{D})$, we put

$$\{\varphi_1\} = \varphi_1, \tag{1.4}$$

$$\{\varphi_1, \dots, \varphi_r\} = G_D^\ell(q_r \varphi_1 \dots \varphi_r) \quad \text{for } r > 1 \tag{1.5}$$

We prove:

THEOREM 1.1. *Suppose that $f_0, f_1, f_2, \dots \in \mathcal{B}(\bar{D})$ and let $0 < \beta \leq f_0(x) \leq \gamma$ for all $x \in \bar{D}$ where β and γ are constants. Put $\varphi_i = K_D^\ell f_i$. The functions*

$$z_C(x) = \mathbb{N}_x e^{-\langle f_0, X_D \rangle} \prod_{i \in C} \langle f_i, X_D \rangle, \quad x \in D \tag{1.6}$$

can be evaluated by the recursive formulae

$$\begin{aligned} z_C &= \varphi_i \quad \text{for } C = \{i\}, \\ z_C &= \sum_{2 \leq r \leq |C|} \sum_{\mathfrak{P}_r(C)} \{z_{C_1}, \dots, z_{C_r}\} \quad \text{for } |C| > 1. \end{aligned} \tag{1.7}$$

THEOREM 1.2. *In notation of Theorem 1.1,*

$$P_\mu e^{-\langle f_0, X_D \rangle} \prod_{i \in C} \langle f_i, X_D \rangle = e^{-\langle V_D(f_0), \mu \rangle} \sum_{\mathfrak{P}(C)} \langle z_{C_1}, \mu \rangle \dots \langle z_{C_r}, \mu \rangle \tag{1.8}$$

for every $\mu \in \mathcal{M}_c(D)$.

Theorems 1.1 and 1.2 imply the following expressions:

$$P_x \langle f, X_D \rangle = \mathbb{N}_x \langle f, X_D \rangle = K_D f(x), \tag{1.9}$$

(1.10)
$$P_x \langle f, X_D \rangle^2 = \mathbb{N}_x \langle f, X_D \rangle^2 + [\mathbb{N}_x \langle f, X_D \rangle]^2 = G_D[q_2 (K_D f)^2](x) + [K_D f(x)]^2.$$

1.2. Preparations. Let $\mathcal{D}_i = \frac{\partial}{\partial \lambda_i}$. Suppose that $F^\lambda(x)$ is a function of $x \in \bar{D}$ and $\lambda = \{\lambda_1, \lambda_2, \dots\} \in [0,1]^\infty$ which depends only on a finite number of λ_i. Put $F \in \mathbb{C}^\infty$ if F and all its partials with respect to λ are bounded. Write $\mathcal{D}_C F$ for $\mathcal{D}_{i_1} \dots \mathcal{D}_{i_r} F$ if $C = \{i_1 < \dots < i_r\}$. [1] Let

$$y_C^\lambda = f_0 + \sum_{i \in C} \lambda_i f_i,$$

$$Y_C^\lambda = Y_0 + \sum_{i \in C} \lambda_i Y_i$$

where $Y_i = \langle f_i, X_D \rangle$.

LEMMA 1.1. *Suppose that for all x, $f_0(x) \geq \beta > 0$ and $f_i(x) < \gamma$ for $i \in C$. Then the functions*

$$u_C^\lambda(x) = \mathbb{N}_x(1 - e^{-Y_C^\lambda}) = V_D(y_C^\lambda)(x) \tag{1.11}$$

belong to $\mathbb{C}^\infty$ and

$$z_C = (-1)^{|C|+1} (\mathcal{D}_C u_C^\lambda)|_{\lambda=0}. \tag{1.12}$$

PROOF. 1°. Put $I = \langle 1, X_D \rangle$. First, we prove a bound [2]

$$\mathbb{N}_x I \leq 1. \tag{1.13}$$

Note that by **4**.(1.1), **3**.(2.6) and **3**.(2.1),

$$\mathbb{N}_x(1 - e^{-\lambda I}) = -\log P_x e^{-\lambda I} = V_D(\lambda)(x) \leq K_D(\lambda)(x) = \lambda. \tag{1.14}$$

Since $(1 - e^{-\lambda I})/\lambda \to I$ as $\lambda \downarrow 0$, (1.13) follows from (1.14) by Fatou's lemma.

2°. For every $\beta > 0$ and every $n \geq 1$, the function $\varphi_n(t) = e^{-\beta t} t^{n-1}$ is bounded on $\mathbb{R}_+$. Note that $Y_i \leq \gamma I$ for $i \in C$ and $e^{-Y_C^\lambda} \leq e^{-\beta I}$. Therefore

$$|\mathcal{D}_{i_1} \dots \mathcal{D}_{i_n} (1 - e^{-Y_C^\lambda})| = Y_{i_1} \dots Y_{i_n} e^{-Y_C^\lambda} \leq \gamma^n I \varphi_n(I) \leq \text{const.}\, I.$$

It follows from (1.11) and (1.13) that, for all $i_1, \dots, i_n \in C$,

$$\mathcal{D}_{i_1} \dots \mathcal{D}_{i_n} u_C^\lambda = \mathbb{N}_x \mathcal{D}_{i_1} \dots \mathcal{D}_{i_n} (1 - e^{-Y_C^\lambda}).$$

Hence $u_C^\lambda \in \mathbb{C}^\infty$ and it satisfies (1.12). □

1.3. Proof of Theorem 1.1. 1°. It is sufficient to prove (1.6) for bounded $f_1, f_2, \dots$. (This restriction can be removed by a monotone passage to the limit.) Operators K_D, G_D, K_D^ℓ and G_D^ℓ map bounded functions to bounded functions. Indeed, if $0 \leq f \leq \gamma$, then $K_D^\ell f \leq K_D f \leq \gamma$ and $G_D^\ell f \leq G_D f \leq \gamma \Pi_x \tau_D$ and, for a bounded set D, $\Pi_x \tau_D$ is bounded by Proposition **3**.1.1.

[1] Put $\mathcal{D}_C F = F$ for $C = \emptyset$.

[2] After we prove Theorem 1.1, a stronger version of (1.13) $\mathbb{N}_x I = 1$ will follow from (1.9).

2°. Let $F \in \mathbb{C}^\infty$. We write $F \sim 0$ if $\mathcal{D}_C F|_{\lambda=0} = 0$ for all sets C (including the empty set). Clearly, $F \sim 0$ if, for some $n \geq 1$,

$$F^\lambda = \sum_1^n \lambda_i^2 Q_i^\lambda + |\lambda|^n \varepsilon^\lambda \tag{1.15}$$

where $|\lambda| = \sum_1^n \lambda_i$, Q_i^λ are polynomials in λ with coefficients that are bounded Borel functions in x and ε^λ is a bounded function tending to 0 at each x as $|\lambda| \to 0$. It follows from Taylor's formula that, if $F \sim 0$, then F can be represented in the form (1.15) with every $n \geq 1$. We write $F_1 \sim F_2$ if $F_1 - F_2 \sim 0$. Note that, if $F \sim 0$, then $F\tilde{F} \sim 0$ for every $\tilde{F} \in \mathbb{C}^\infty$ and therefore $F_1 F_2 \sim \tilde{F}_1 \tilde{F}_2$ if $F_1 \sim \tilde{F}_1$ and $F_2 \sim \tilde{F}_2$. Operators K_D, G_D, K_D^ℓ and G_D^ℓ preserve the relation $\sim$.

Put $u^\lambda = u_C^\lambda$. It follows from Lemma 1.1 that

$$u^\lambda \sim u^0 + \sum_B (-1)^{|B|-1} \lambda_B z_B \tag{1.16}$$

where B runs over all nonempty subsets of C.

3°. By **3**(2.8), **3**(2.11) and Taylor's formula, for every n,

$$\begin{aligned} \psi(u^\lambda) = \psi(u^0) &+ \psi_1(u^0)(u^\lambda - u^0) \\ &+ \sum_{r=2}^n \frac{1}{r!} \psi_r(u^0)(u^\lambda - u^0)^r + R_\lambda (u^\lambda - u^0)^n \end{aligned} \tag{1.17}$$

where

$$R_\lambda(x) = \frac{1}{n!} \int_0^\infty t^n (e^{-\lambda\theta} - e^{-\lambda u^0}) N(x, dt)$$

with θ between u^0 and u^λ. By (1.16),

$$\begin{aligned} (u^\lambda - u^0)^r &\sim \sum_{B_1, \dots, B_r} \prod_1^r (-1)^{|B_i|-1} \lambda_{B_i} z_{B_i} \\ &= r! \sum_B (-1)^{|B|-r} \lambda_B \sum_{\mathfrak{P}_r(B)} z_{B_1} \dots z_{B_r}. \end{aligned} \tag{1.18}$$

Since $u^0 = V_D(f_0)$ and, by (1.1), $\psi_1(u^0) = \ell$, we conclude from (1.17), (1.18) and (1.3) that

$$\psi(u^\lambda) \sim \psi(u^0) + \ell \sum_1^n \lambda_i z_i + \ell \sum_{|B| \geq 2} (-1)^{|B|-1} \lambda_B z_B + \sum_{B \subset C} (-1)^{|B|} \rho_B$$

where $\rho_B = 0$ for $|B| = 1$ and

$$\rho_B = \sum_{r \geq 2} q_r \sum_{\mathfrak{P}_r(B)} z_{B_1} \dots z_{B_r} \quad \text{for } |B| \geq 2.$$

Hence,
(1.19)

$$G_D[\psi(u^\lambda)] \sim G_D[\psi(u^0)+\ell\sum_1^n \lambda_i z_i+\ell\sum_{|B|\ge 2}(-1)^{|B|-1}\lambda_B z_B+\sum_{B\subset C}(-1)^{|B|}\rho_B].$$

By **2**.(2.1) and (1.11), $u^\lambda + G_D\psi(u^\lambda) = K_D y^\lambda$. By using (1.16) and (1.19) and by comparing the coefficients at λ_B, we get

$$z_i + G_D(\ell z_i) = K_D f_i \quad \text{for } i \in C \tag{1.20}$$

and

$$z_B + G_D(\ell z_B) = G_D\rho_B \quad \text{for } |B| \ge 2. \tag{1.21}$$

By Theorem **3**.1.1,

$$z = K_D^\ell f(x)$$

is a unique solution of the integral equation

$$z + G_D(\ell z) = K_D f$$

and

$$\varphi = G_D^\ell \rho$$

is a unique solution of the equation

$$\varphi + G_D(\ell\varphi) = G_D\rho.$$

Therefore the equations (1.20) and (1.21) imply (1.7). □

1.4. Proof of Theorem 1.2. We have

$$\begin{aligned} P_\mu e^{-Y_C^\lambda} &= P_\mu e^{-Y_0}\prod_{i\in C} e^{-\lambda_i Y_i} \sim P_\mu e^{-Y_0}\prod_{i\in C}(1-\lambda_i Y_i) \\ &\sim P_\mu e^{-Y_0} + \sum_{B\subset C}(-1)^{|B|}\lambda_B P_\mu e^{-Y_0} Y_B \end{aligned} \tag{1.22}$$

where $Y_B = \prod_{i\in B} Y_i$ and the sum is taken over nonempty B.

By **4**.(1.1) and **3**.(2.6), $V_D(y_C^\lambda)(x) = \mathbb{N}_x(1-e^{-Y_C^\lambda})$ and therefore, by **3**.(2.7),

$$P_\mu e^{-Y_C^\lambda} = \exp -\langle V_D(y_C^\lambda), \mu\rangle = \exp[-\int_D \mathbb{N}_x(1-e^{-Y_C^\lambda})\mu(dx)]. \tag{1.23}$$

By (1.6), $\mathbb{N}_x e^{-Y_0} Y_B = z_B$ and, since $\mathbb{N}_x(1-e^{-Y_0}) = V_D(f_0)$, we have

$$\begin{aligned} \mathbb{N}_x(1-e^{-Y_C^\lambda}) &= \mathbb{N}_x[1-e^{-Y_0}\prod_{i\in C} e^{-\lambda_i Y_i}] \sim \mathbb{N}_x[1-e^{-Y_0}\prod_{i\in C}(1-\lambda_i Y_i)] \\ &\sim V_D(f_0) - \sum_{B\subset C}(-1)^{|B|}\lambda_B z_B. \end{aligned}$$

Hence,

$$\int_D \mathbb{N}_x(1-e^{-Y_C^\lambda})\mu(dx) \sim \langle V_D(f_0), \mu\rangle - \sum_{B\subset C}(-1)^{|B|}\lambda_B\langle z_B, \mu\rangle. \tag{1.24}$$

This implies

(1.25)
$$\begin{aligned}\exp\{-\int_D \mathbb{N}_x(1-e^{-Y_C^\lambda})\mu(dx)\}\\ &= \exp[-\langle V_D(f_0),\mu)\rangle]\prod_{B\subset C}\exp[(-1)^{|B|}\lambda_B\langle z_B,\mu\rangle]\\ &\sim \exp[-\langle V_D(f_0,\mu)\rangle]\prod_{B\subset C}[1+(-1)^{|B|}\lambda_B\langle z_B,\mu\rangle]\\ &\sim \exp[-\langle V_D(f_0),\mu)\rangle][1+\sum_{B\subset C}(-1)^{|B|}\lambda_B\sum_{\mathfrak{P}(B)}\langle z_{B_1},\mu\rangle\dots\langle z_{B_r},\mu\rangle].\end{aligned}$$

According to (1.23), the left sides in (1.22) and (1.25) coincide. By comparing the coefficients at λ_B in the right sides, we get (1.8). □

2. Diagram description of moments

We deduce from Theorems 1.1 and 1.2 a description of moments in terms of labelled directed graphs.

2.1. Put $f_0 = 1$ and $\ell = \psi'[V_D(1)]$ in formulae (1.6) and (1.7). Suppose that $C = \{i_1,\dots,i_r\}$. The function $z_C(x)$ defined by (1.6) depends on $f_C = \{f_{i_1},\dots,f_{i_r}\}$ which we indicate by writing $z(f_C)$ instead of z_C. In this notation (1.7) takes the form

$$z(f_i) = \varphi_i, \tag{2.1}$$

$$z(f_C) = \sum_{2\le r\le|C|}\sum_{\mathfrak{P}_r(C)}\{z(f_{C_1}),\dots,z(f_{C_r})\}\quad\text{for } |C|>1. \tag{2.2}$$

We consider monomials like $\{\{\varphi_3\varphi_2\}\varphi_1\{\varphi_4\varphi_5\}\}$. There exist one monomial $\{\varphi_1\varphi_2\}$ of degree 2 and four distinguishable monomials of degree 3:

$$\{\varphi_1\varphi_2\varphi_3\},\{\{\varphi_1\varphi_2\}\varphi_3\},\{\{\varphi_2\varphi_3\}\varphi_1\},\{\{\varphi_3\varphi_1\}\varphi_2\}. \tag{2.3}$$

It follows from (2.1) and (2.2) that, for $C = \{i_1,\dots,i_n\}$, $z(f_C)$ is equal to the sum of all monomials of degree n of $\varphi_{i_1},\dots,\varphi_{i_n}$.

Formulae (1.6) and (1.8) imply

$$\mathbb{N}_x e^{-\langle 1,X_D\rangle}\langle f_1,X_D\rangle\dots\langle f_n,X_D\rangle = z(f_1,\dots,f_n)(x)\quad\text{for all } x\in D \tag{2.4}$$

and

(2.5)
$$P_\mu e^{-\langle 1,X_D\rangle}\langle f_1,X_D\rangle\dots\langle f_n,X_D\rangle = e^{-\langle V_D(1),\mu\rangle}\sum_{\mathfrak{P}(C)}\langle z(f_{C_1}),\mu\rangle\dots\langle z(f_{C_r}),\mu\rangle$$

where $C = \{1,\dots,n\}$.

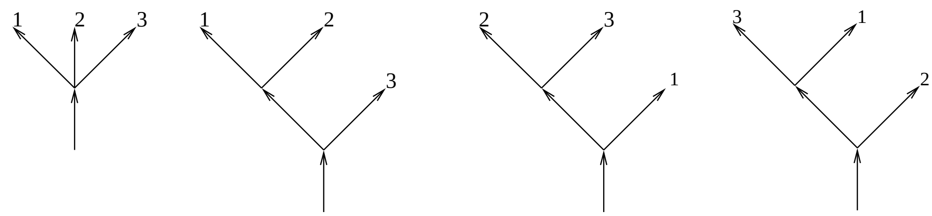

FIGURE 1

2.2. A diagram $D \in \mathbb{D}_n$ is a rooted tree with the leaves marked by $1, 2, \dots, n$. To every monomial of degree n there corresponds $D \in \mathbb{D}_n$. Here are the diagrams corresponding to the monomials (2.3):

Every diagram consists of a set V of vertices (or sites) and a set A of arrows. We write $a : v \to v'$ if v is the beginning and v' is the end of an arrow a. We denote by $a_+(v)$ the number of arrows which end at v and by $a_-(v)$ the number of arrows which begin at v. Note that $a_+(v) = 0, a_-(v) = 1$ for the root and $a_+(v) = 1, a_-(v) = 0$ for leaves.

We label each site v of $D \in \mathbb{D}_n$ by a variable y_v. We take $y_v = x$ for the root v and $y_v = z_i$ for the leaf i. We also label every arrow $a : v \to v'$ by a kernel $r_a(y_v, dy_{v'})$. Here r_a is one of two kernels corresponding to the operators G^ℓ_D and K^ℓ_D by the formulae

$$G^\ell_D f(x) = \int_D g^\ell_D(x, dy) f(y)$$

and

$$K^\ell_D f(x) = \int_{\partial D} k^\ell_D(x, dy) f(y).$$

More precisely, if $a = v \to v'$, then $r_a = g^\ell_D(y_v, dy_{v'})$ if v, v' are not leaves and $r_a = k^\ell_D(y_v, dz_i)$ if v' is a leaf i. We associate with $D \in \mathbb{D}_n$ a function (2.6)

$$z^D(f_1, \dots, f_n) = \int \prod_{a \in A} r_a(y_v, dy_{v'}) \prod_{v \in V} q_{a_-(v)}(y_v) \prod_1^n k^\ell_D(y_{v_i}, dz_i) f_i(z_i)$$

where v_i is the beginning of the arrow with the end at a leaf i. [3]

EXAMPLES. For the first diagram on Figure 1,

$$z^D(f_1, f_2, f_3) \\ = \int g^\ell_D(x, dy) q_3(y) k^\ell_D(y, dz_1) f_1(z_1) k^\ell_D(y, dz_2) f_2(z_2) k^\ell_D(y, dz_3) f_3(z_3).$$

[3] We put $q_0 = 1$ to serve leaves v for which $a_-(v) = 0$.

For the second diagram,

$$z^D(f_1,f_2,f_3) = \int g_D^\ell(x,dy_1)q_2(y_1)k_D^\ell(y_1,dz_3)f_3(z_3) \\ g_D^\ell(y_1,dy_2)q_2(y_2)k_D^\ell(y_2,dz_1)f_1(z_1)k_D^\ell(y_2,dz_2)f_2(z_2).$$

We note that

$$z(f_1,\dots,f_n) = \sum_{D\in\mathbb{D}_n} z^D(f_1,\dots,f_n). \tag{2.7}$$

3. Absolute continuity results

3.1. In this section we prove:

THEOREM 3.1. *Let D be a bounded domain of class $C^{2,\lambda}$ and let γ be the surface area on ∂D. For every Borel subset B of $(\partial D)^n$,*

$$\mathbb{N}_x e^{-\langle 1,X_D\rangle}\int_B X_D(dy_1)\dots X_D(dy_n) \\ = \int_B \rho^x(y_1,\dots,y_n)\gamma(dy_1)\dots\gamma(dy_n) \tag{3.1}$$

with a strictly positive ρ^x .

For every $\mu\in\mathcal{M}_c(D)$,

$$P_\mu e^{-\langle 1,X_D\rangle}\int_B X_D(dy_1)\dots X_D(dy_n) \\ = e^{-\langle V_D(1),\mu\rangle}\int_B \rho^\mu(y_1,\dots,y_n)\gamma(dy_1)\dots\gamma(dy_n) \tag{3.2}$$

with a strictly positive ρ^μ.

Theorem 3.1 implies that the class of null sets for each of measures (0.2) and (0.3) (we call them the *moment measures*) coincides with the class of null sets for the measure γ^n. In other words, all these measures are equivalent.

THEOREM 3.2. *Suppose $A\in\mathcal{F}_{\supset D}$. Then either $P_\mu(A)=0$ for all $\mu\in\mathcal{M}_c(D)$ or $P_\mu(A)>0$ for all $\mu\in\mathcal{M}_c(D)$.*

If $A\in\mathcal{F}_\partial$, then either $P_\mu(A)=0$ for all $\mu\in\mathcal{M}_c(E)$ or $P_\mu(A)>0$ for all $\mu\in\mathcal{M}_c(E)$.

3.2. Proof of Theorem 3.1. It is sufficient to prove that formulae (3.1) and (3.2) hold for $B=B_1\times\dots\times B_n$ where $B_1,\dots,B_n$ are Borel subsets of ∂D. If we demonstrate that

$$z^D(f_1,\dots,f_n) = \int \rho^D(y_1,\dots,y_n)f_1(y_1)\dots f_n(y_n)\gamma(dy_1)\dots\gamma(dy_n) \tag{3.3}$$

with $\rho^D>0$ for $f_1=1_{B_1},\dots,f_n=1_{B_n}$, then (3.1) and (3.2) will follow from (2.4), (2.5) and (2.7). For a domain D of class $C^{2,\lambda}$, $k_D^\ell(x,dy)=k_D^\ell(x,y)\gamma(dy)$ where $k_D^\ell(x,y)$ is the Poisson kernel for $Lu-\ell u$. Since $k_D^\ell(x,y)>0$, formula (2.6) implies (3.3). □

To prove Theorem 3.2 we need some preparations.

3.3. Poisson random measure.

THEOREM 3.3. *Suppose that $\mathcal{R}$ is a finite measure on a measurable space $(S, \mathcal{B})$. Then there exists a random measure (Y, Q) on S with the properties:*
(a) $Y(B_1), \dots, Y(B_n)$ are independent for disjoint $B_1, \dots, B_n$;
(b) $Y(B)$ is a Poisson random variable with the mean $\mathcal{R}(B)$, i.e.,

$$Q\{Y(B) = n\} = \frac{1}{n!}\mathcal{R}(B)^n e^{-\mathcal{R}(B)} \quad \text{for } n = 0, 1, 2, \dots.$$

For every function $F \in \mathcal{B}$,

$$Qe^{-\langle F,Y\rangle} = \exp[-\int_S (1 - e^{-F(z)})\mathcal{R}(dz)]. \tag{3.4}$$

PROOF. Consider independent identically distributed random elements $Z_1, \dots, Z_n, \dots$ of S with the probability distribution $\bar{\mathcal{R}}(B) = \mathcal{R}(B)/\mathcal{R}(S)$. Let N be the Poisson random variable with mean value $\mathcal{R}(S)$ independent of $Z_1, \dots, Z_n, \dots$. Put $Y(B) = 1_B(Z_1) + \dots + 1_B(Z_N)$. Note that $Y = \delta_{Z_1} + \dots + \delta_{Z_N}$ where δ_z is the unit measure concentrated at z. Therefore $\langle F, Y\rangle = \sum_1^N F(Z_i)$ and (3.4) follows from the relation

$$Qe^{-\langle F,Y\rangle} = \sum_{m=0}^{\infty} \frac{1}{m!}\mathcal{R}(S)^m e^{-\mathcal{R}(S)} \prod_1^m Qe^{-F(Z_i)}.$$

By taking $F = \lambda 1_B$ we get

$$Qe^{-\lambda Y(B)} = \exp[-(1 - e^{-\lambda})\mathcal{R}(B)]$$

which implies the property (b). If $B_1, \dots, B_n$ are disjoint, then, by applying (3.4) to $F = \sum_1^n \lambda_i 1_{B_i}$, we get

$$Qe^{-\sum \lambda_i Y(B_i)} = e^{-\sum(1-e^{-\lambda_i})\mathcal{R}(B_i)} = \prod Qe^{-\lambda_i Y(B_i)}$$

which implies (a). □

We conclude from (3.4) that (Y, Q) is an infinitely divisible random measure. It is called *the Poisson random measure with intensity* $\mathcal{R}$. This is an integer-valued measure concentrated on a finite random set.

3.4. Poisson representation of infinitely divisible measures.

THEOREM 3.4. *Let (X, P) be an infinitely divisible measure on a measurable Luzin space E with the canonical measure $\mathcal{R}$. Consider the Poisson random measure (Y, Q) on $S = \mathcal{M}(E)$ with intensity $\mathcal{R}$ and put $\tilde{X}(B) = \int_{\mathcal{M}} \nu(B)Y(d\nu)$. The random measure $(\tilde{X}, Q)$ has the same probability distribution as (X, P) and, for every $F \in \mathcal{B}_{\mathcal{M}}$, we have*

$$PF(X) = Q\langle F, Y\rangle = \sum_0^{\infty} \frac{1}{n!} e^{-\mathcal{R}(\mathcal{M})} \int \mathcal{R}(d\nu_1) \dots \mathcal{R}(d\nu_n) F(\nu_1 + \dots + \nu_n). \tag{3.5}$$

PROOF. Note that $\langle f,\tilde{X}\rangle = \langle F,Y\rangle$ where $F(\nu)=\langle f,\nu\rangle$. By (3.4), we get

$$Qe^{-\langle f,\tilde{X}\rangle} = Qe^{-\langle F,Y\rangle} = \exp\left[-\int_{\mathcal{M}}(1-e^{-\langle f,\nu\rangle})\mathcal{R}(d\nu)\right].$$

This implies the first part of the theorem. The second part follows from the expression $Y(B)=1_B(Z_1)+\dots+1_B(Z_N)$ for Y introduced in the proof of Theorem 3.3. □

3.5. Proof of Theorem 3.2. 1°. By applying Theorem 3.4 to the random measure (P_μ, X_D) and a function $e^{-\langle 1,\nu\rangle}F(\nu)$ we get

$$(3.6)\quad P_\mu e^{-\langle 1,X_D\rangle}F(X_D) = \sum_0^\infty \frac{1}{n!} Z_D(\mu)\int \mathcal{R}^*_\mu(d\nu_1)\dots\mathcal{R}^*_\mu(d\nu_n)F(\nu_1+\dots+\nu_n)$$

where

$$(3.7)\quad Z_D(\mu)=e^{-\mathcal{R}_\mu[\mathcal{M}(D)]},\quad \text{and } \mathcal{R}^*_\mu(d\nu)=e^{-\langle 1,\nu\rangle}\mathcal{R}_\mu(d\nu).$$

2°. Let F be a positive measurable function on $\mathcal{M}(\partial D)$ and let

$$f^n(x_1,\dots,x_n)=\int F(\nu_1+\dots+\nu_n)\mathcal{R}^*_{x_1}(d\nu_1)\dots\mathcal{R}^*_{x_n}(d\nu_n).$$

We prove that, if $\tilde{D}\Subset D$ and $\mu\in\mathcal{M}_c(\tilde{D})$, then $F(X_D)=0$ P_μ-a.s. if and only if

$$(3.8)\quad \int f^n(x_1,\dots,x_n)\gamma_{\tilde{D}}(dx_1)\dots\gamma_{\tilde{D}}(dx_n)=0\quad\text{for all } n.$$

Indeed, by the Markov property of X,

$$(3.9)\quad P_\mu e^{-\langle 1,X_D\rangle}F(X_D)=P_\mu P_{X_{\tilde{D}}}e^{-\langle 1,X_D\rangle}F(X_D).$$

By (3.6) and (3.9),

$$(3.10)\quad P_\mu e^{-\langle 1,X_D\rangle}F(X_D) = \sum_{n=0}^\infty \frac{1}{n!}P_\mu Z_D(X_{\tilde{D}})\int X_{\tilde{D}}(dx_1)\dots X_{\tilde{D}}(dx_n)f^n(x_1,\dots,x_n).$$

Since $Z_D(X_{\tilde{D}})>0$, the condition $F(X_D)=0$ P_μ-a.s. is equivalent to the condition: for every n,

$$(3.11)\quad \int X_{\tilde{D}}(dx_1)\dots X_{\tilde{D}}(dx_n)f^n(x_1,\dots,x_n)=0\quad P_\mu\text{-a.s.}$$

It follows from Theorem 3.1 that the condition (3.11) is equivalent to the condition (3.8).

3°. Suppose μ_1 and μ_2 belong to $\mathcal{M}_c(D)$. There exists $\tilde{D}\Subset D$ which contains supports of μ_1 and μ_2. By 2°, $F(X_D)=0$ P_{μ_1}-a.s. if and only if $F(X_D)=0$ P_{μ_2}-a.s. If $A\in\mathcal{F}_{\supset D}$, then by the Markov property of X,

$$P_{\mu_i}(A)=P_{\mu_i}F(X_D)$$

where $F(\nu) = P_\nu(A)$. This implies the first statement of Theorem 3.2.

If $\mu_1, \mu_2 \in \mathcal{M}_c(E)$, then $\mu_1, \mu_2 \in \mathcal{M}_c(D)$ for a domain of class $C^{2,\lambda}$ such that $D \Subset E$. If $A \in \mathcal{F}_\partial$, then $A \in \mathcal{F}_{\supset D}$ and the second part of Theorem 3.2 follows from the first one. □

4. Notes

4.1. The results of the first two sections are applicable to all (ξ, ψ)-superprocesses described in Section **3**.2.2, and the proofs do not need any modification. The absolute continuity results can be extended to (ξ, ψ)-superprocesses under an additional assumption that the Martin boundary theory is applicable to ξ. [4] The boundary ∂E and the Poisson kernel are to be replaced by the Martin boundary and the Martin kernel. The role of the surface area is played by the measure corresponding to the harmonic function $h = 1$.

4.2. A diagram description of moments of higher order was given, first, in [Dy88]. There only $\psi(u) = u^2$ was considered. In [Dy91b] the moments of order n were evaluated under the assumption that ψ of the form **3**.(2.8) has a bounded continuous derivative $\frac{d^n\psi}{du^n}$. [See also [Dy04a].] Brief description of these results is given on pages 201–203 of [**D**]. [5] The main recent progress is the elimination of the assumption about differentiability of ψ which allows to cover the case $\psi(u) = u^\alpha, 1 < \alpha < 2$.

4.3. The first absolute continuity results for superprocesses were obtained in [EP91]. Let (X_t, P_μ) be a (ξ, ψ)-superprocess with $\psi(u) = u^2/2$. To every $\mu \in \mathcal{M}(E)$ there correspond measures p_t^μ on E and measures Q_t^μ on $\mathcal{M}(E)$ defined by the formulae

$$p_t^\mu(B) = \int \mu(dx)\Pi_x\{\xi_t \in B\},$$
$$Q_t^\mu(C) = P_\mu\{X_t \in C\}.$$

Let $h > 0$. Evans and Perkins proved that $Q_t^{\mu_1}$ is absolutely continuous with respect to $Q_{t+h}^{\mu_2}$ for all $t > 0$ if and only if $p_t^{\mu_1}$ is absolutely continuous with respect to $p_{t+h}^{\mu_2}$ for all $t > 0$.

Independently, Mselati established an absolute continuity property for the excursion measures $\mathbb{N}_x$ of the Brownian snake: if C belongs to the σ-algebra generated by the stochastic values of all subsolutions and supersolutions of the equation $\Delta u = u^2$, then, for every $x_1, x_2 \in E$, $\mathbb{N}_{x_1}(C) = 0$

[4]The key condition – the existence of a Green's function – is satisfied for L-diffusions in a wide class of the so-called Greenian domains. The Martin boundary theory for such domains can be found in Chapter 7 of [**D**].

[5]Figure 1.2 is borrowed from page 202 in [**D**]. We also corrected a few misprints in formulae which could confuse a reader.[For instance the value of q_m on pages 201–203 must be multiplied by $(-1)^m$.]

if $\mathbb{N}_{x_2}(C) = 0$. (See Proposition 2.3.5 in [Ms02a] or Proposition 2.18 in [Ms04].)

A proof of Theorem 3.2 is given in [Dy04c]. The case of infinitely differentiable ψ was considered earlier in [Dy04a], Theorem 6.2.

CHAPTER 6

Poisson capacities

A key part of the proof that all solutions of the equation $\Delta u = u^\alpha$ are σ-moderate is establishing bounds for w_Γ and u_Γ in terms of a capacity of Γ. In the case $\alpha = 2$, Mselati found such bounds by using Cap^∂ introduced by Le Gall. This kind of capacity is not applicable for $\alpha \neq 2$. We replace it by a family of Poisson capacities. In this chapter we establish relations between these capacities which will be used in Chapters **8** and **9**.

The Poisson capacities are a special case of (k,m)-capacities described in Section 1.

1. Capacities associated with a pair (k, m)

1.1. Three definitions of (k,m)-capacities. Fix $\alpha > 1$. Suppose that $k(x,y)$ is a positive lower semicontinuous function on the product $E \times \tilde{E}$ of two separable locally compact metric spaces and m is a Radon measure on E. A (k,m)-capacity is a Choquet capacity on $\tilde{E}$. We give three equivalent definitions of this capacity.

Put

$$(1.1)\quad (K\nu)(x) = \int_{\tilde{E}} k(x,y)\nu(dy), \quad \mathcal{E}(\nu) = \int_E (K\nu)^\alpha dm \quad \text{for } \nu \in \mathcal{M}(\tilde{E})$$

and

$$(1.2)\qquad \hat{K}(f)(y) = \int_E m(dx) f(x) k(x,y) \quad \text{for } f \in \mathcal{B}(E).$$

Define $\mathrm{Cap}(\Gamma)$ for subsets Γ of $\tilde{E}$ by one of the following three formulae:

$$(1.3)\qquad \mathrm{Cap}(\Gamma) = \sup\{\mathcal{E}(\nu)^{-1} : \nu \in \mathcal{P}(\Gamma)\},$$

$$(1.4)\qquad \mathrm{Cap}(\Gamma) = [\sup\{\nu(\Gamma) : \nu \in \mathcal{M}(\Gamma), \mathcal{E}(\nu) \leq 1\}]^\alpha,$$

$$(1.5)\qquad \mathrm{Cap}(\Gamma) = [\inf\{\int_E f^{\alpha'} dm : f \in \mathcal{B}(E), \hat{K}f \geq 1 \quad \text{on } \Gamma\}]^{\alpha-1}$$

where $\alpha' = \alpha/(\alpha-1)$. We refer to [AH96], Chapter 2 for the proof that the $\mathrm{Cap}(\Gamma)$ defined by (1.4) or by (1.5) satisfies the conditions **2**.4.A, **2**.4.B and **2**.4.C and therefore all Borel subsets are capacitable. [1] [The equivalence of (1.4) and (1.5) is proved also in [**D**], Theorem 13.5.1.]

[1] In [AH96] a wider class of kernels is considered. The result is stated for the case $E = \mathbb{R}^d$ but no specific property of $\mathbb{R}^d$ is used in the proofs.

To prove the equivalence of (1.3) and (1.4), we note that $\nu \in \mathcal{M}(\Gamma)$ is equal to $t\mu$ where $t = \nu(\Gamma)$ and $\mu = \nu/t \in \mathcal{P}(\Gamma)$ and

$$\sup_{\nu \in \mathcal{M}(\Gamma)} \{\nu(\Gamma) : \mathcal{E}(\nu) \leq 1\} = \sup_{\mu \in \mathcal{P}(\Gamma)} \sup_{t \geq 0} \{t : t^\alpha \mathcal{E}(\mu) \leq 1\} = \sup_{\mu \in \mathcal{P}(\Gamma)} \mathcal{E}(\mu)^{-1/\alpha}.$$

2. Poisson capacities

In this chapter we deal with a special type of (k, m)-capacities associated with the Poisson kernel $k = k_E$ for an operator L. The function $k_E(x, y)$ is continuous on $E \times \tilde{E}$ where E is a $C^{2,\lambda}$-domain in $\mathbb{R}^d$ and $\tilde{E} = \partial E$. We use notation Cap for the Poisson capacity corresponding to

$$m(dx) = \rho(x)dx \quad \text{with } \rho(x) = d(x, \partial E) \tag{2.1}$$

and we denote by Cap_x the Poisson capacity corresponding to

$$m(dy) = g_E(x, y)dy \tag{2.2}$$

where g_E is the Green function in E for L. [In the case of Cap_x, $\mathcal{E}(\nu)$ has to be replaced by

$$\mathcal{E}_x(\nu) = \int_E g_E(x, y) h_\nu(y)^\alpha dy = [G_E(K\nu)^\alpha](x)$$

in formulae (1.3)–(1.4).]

2.1. Results.

An upper bound of $\mathrm{Cap}(\Gamma)$ is given by:

THEOREM 2.1. *For all $\Gamma \in \mathcal{B}(\partial E)$,*

$$\mathrm{Cap}(\Gamma) \leq C \,\mathrm{diam}(\Gamma)^{\gamma_+} \tag{2.3}$$

where

$$\gamma = d\alpha - d - \alpha - 1 \quad \textit{and } \gamma_+ = \gamma \vee 0. \tag{2.4}$$

The second theorem establishes a lower bound for Cap_x in terms of Cap.

The values $\alpha < (d+1)/(d-1)$ are called *subcritical* and the values $\alpha \geq (d+1)/(d-1)$ are called *supercritical*.

THEOREM 2.2. *Suppose that L is an operator of divergence form **1**.(4.2) and $d \geq 3$. Put*

$$\varphi(x, \Gamma) = \rho(x) d(x, \Gamma)^{-d}. \tag{2.5}$$

If α is subcritical, then there exists a constant $C > 0$ such that

$$\mathrm{Cap}_x(\Gamma) \geq C \varphi(x, \Gamma)^{-1} \,\mathrm{Cap}(\Gamma). \tag{2.6}$$

for all Γ and x.

If α is supercritical, then, for every $\kappa > 0$ there exists a constant $C_\kappa > 0$ such that

$$\mathrm{Cap}_x(\Gamma) \geq C_\kappa \varphi(x, \Gamma)^{-1} \,\mathrm{Cap}(\Gamma) \tag{2.7}$$

for all Γ and x subject to the condition

$$d(x, \Gamma) \geq \kappa \,\mathrm{diam}(\Gamma). \tag{2.8}$$

3. Upper bound for Cap(Γ)

To prove Theorem 2.1 we use the straightening of the boundary described in Section 4.2 of the Introduction. As the first step, we consider a capacity on the boundary $E_0 = \{x = (x_1, \dots, x_d) : x_d = 0\}$ of a half-space $E_+ = \{x = (x_1, \dots, x_d) : x_d > 0\} = \mathbb{R}^{d-1} \times (0, \infty)$.

3.1. Capacity $\widetilde{\mathrm{Cap}}$. Put

$$\begin{aligned} r(x) &= d(x, E_0) = x_d, \\ \mathbb{E} &= \{x = (x_1, \dots, x_d) : 0 < x_d < 1\}, \\ \tilde{k}(x, y) &= r(x)|x - y|^{-d}, x \in \mathbb{E}, y \in E_0 \end{aligned} \tag{3.1}$$

and consider a measure

$$\tilde{m}(dx) = r(x)dx \tag{3.2}$$

on $\mathbb{E}$. Denote by $\widetilde{\mathrm{Cap}}$ the $(\tilde{k}, \tilde{m})$-capacity on E_0.

Note that

$$\tilde{k}(x/t, y/t) = t^{d-1}\tilde{k}(x, y) \quad \text{for all } t > 0.$$

To every $\nu \in \mathcal{P}(E_0)$ there corresponds a measure $\nu_t \in \mathcal{P}(E_0)$ defined by the formula $\nu_t(B) = \nu(tB)$. We have

$$\int_{E_0} f(y)\nu_t(dy) = \int_{E_0} f(y/t)\nu(dy)$$

for every function $f \in \mathcal{B}(E_0)$. Put $\tilde{h}_\nu = \tilde{K}\nu$. Note that

$$\tilde{h}_{\nu_t}(x/t) = \int_{E_0} \tilde{k}(x/t, y)\nu_t(dy) = \int_{E_0} \tilde{k}(x/t, y/t)\nu(dy) = t^{d-1}\tilde{h}_\nu(x). \tag{3.3}$$

Change of variables $x = t\tilde{x}$ and (3.3) yield

$$\tilde{\mathcal{E}}(\nu_t) = t^\gamma \tilde{\mathcal{E}}(\nu, t\mathbb{E})$$

where

$$\tilde{\mathcal{E}}(\nu) = \int_{\mathbb{E}} \tilde{h}_\nu^\alpha \, d\tilde{m}, \quad \tilde{\mathcal{E}}(\nu, B) = \int_B \tilde{h}_\nu^\alpha \, d\tilde{m}$$

for $B \in \mathcal{B}(E_+)$ and γ defined by (2.4).

If $t \geq 1$, then $t\mathbb{E} \supset \mathbb{E}$ and we have

$$\tilde{\mathcal{E}}(\nu_t) \geq t^\gamma \tilde{\mathcal{E}}(\nu). \tag{3.4}$$

LEMMA 3.1. *If* $\mathrm{diam}(\Gamma) \leq 1$, *then*

$$\widetilde{\mathrm{Cap}}(\Gamma) \leq C_d(\mathrm{diam}(\Gamma))^\gamma. \tag{3.5}$$

The constant C_d depends only on the dimension d. (It is equal to $\widetilde{\mathrm{Cap}}(U)$ where $U = \{x \in E_0 : |x| \leq 1\}$.

PROOF. Since $\widetilde{\mathrm{Cap}}$ is translation invariant, we can assume that $0 \in \Gamma$. Let $t = \mathrm{diam}(\Gamma)^{-1}$. Since $t\Gamma \subset U$, we have

$$\widetilde{\mathrm{Cap}}(t\Gamma) \le \widetilde{\mathrm{Cap}}(U). \tag{3.6}$$

Since $\nu \to \nu_t$ is a 1-1 mapping from $\mathcal{P}(t\Gamma)$ onto $\mathcal{P}(\Gamma)$, we get

$$\widetilde{\mathrm{Cap}}(\Gamma) = \sup_{\nu_t \in \mathcal{P}(\Gamma)} \tilde{\mathcal{E}}(\nu_t)^{-1} = \sup_{\nu \in \mathcal{P}(t\Gamma)} \tilde{\mathcal{E}}(\nu_t)^{-1}.$$

Therefore, by (3.4) and (1.3),

$$\widetilde{\mathrm{Cap}}(\Gamma) \le t^{-\gamma}\widetilde{\mathrm{Cap}}(t\Gamma)$$

and (3.6) implies (3.5). □

3.2. Three lemmas.

LEMMA 3.2. *Suppose that E_1, E_2, E_3 are bounded domains, E_1 and E_2 are smooth and $\bar{E}_2 \subset E_3$. Then there exists a smooth domain D such that*

$$E_1 \cap E_2 \subset D \subset E_1 \cap E_3. \tag{3.7}$$

PROOF. The domain $D_0 = E_1 \cap E_2$ is smooth outside $L = \partial E_1 \cap \partial E_2$. We get D by a finite number of small deformations of D_0 near L. Let $q \in L$ and let U be the ε-neighborhood of q. Consider coordinates $(y_1, y_2, \dots, y_d)$ in U and put $y = (y_1, \dots, y_{d-1}), r = y_d$. If ε is sufficiently small, then the coordinate system can be chosen in which the intersections of E_i with U are described by the conditions $r < f_i(y)$ where f_1, f_2, f_3 are smooth functions. There exists an infinitely differentiable function $a(r)$ such that $r \wedge 0 \le a(r) \le r \wedge \varepsilon$ and $a(r) = 0$ for $r > \varepsilon/2$. Put $g = f_2 + a(f_1 - f_2)$ and replace the part of D_0 in U by $\{(y, r) : r < g(y)\}$ without changing the part outside U. Since $g \ge f_1 \wedge f_2$, we get a domain D_1 which contains D_0. Since $g \le f_1 \wedge (f_2 + \varepsilon)$, D_1 is contained in E_3 if ε is sufficiently small. Finally, the portion of ∂D_1 in U is smooth. After a finite number of deformations of this kind we get a smooth domain which satisfies the condition (3.7). □

LEMMA 3.3. *Suppose $E \subset \mathbb{E}$, $0 \in \Gamma \subset \partial E \cap E_0$ and put $A = \mathbb{E} \setminus E$, $B_\lambda = \{x \in \mathbb{E} : |x| < \lambda\}$. If $d(\Gamma, A) > 2\lambda$, then $B_\lambda \subset E$ and $r(x) = \rho(x)$ for $x \in B_\lambda$.*

PROOF. If $x \in B_\lambda$, then $r(x) \le |x| < \lambda$. If $x \in B_\lambda$ and $y \in A$, then $|x - y| \ge |y| - |x| > \lambda$ because $|y| \ge d(y, \Gamma) \ge d(A, \Gamma) > 2\lambda$. Hence $d(x, A) \ge \lambda$ which implies that $B_\lambda \subset E$.

For $x \in E$, $\rho(x) = d(x, E^c), r(x) = d(x, E_+^c)$ and therefore $\rho(x) \le r(x)$. Put $A_1 = \partial E \cap A, A_2 = \partial E \cap E_0$. For every $x \in E$, $d(x, A_1) = d(x, A), d(x, A_2) \ge r(x)$ and $\rho(x) = d(x, A_1) \wedge d(x, A_2) \ge d(x, A) \wedge r(x)$. If $x \in B_\lambda$, then $r(x) < \lambda \le d(x, A)$ and therefore $\rho(x) \ge r(x)$. Hence $\rho(x) = r(x)$. □

LEMMA 3.4. *There exists a constant $C_\lambda > 0$ such that*

$$\tilde{\mathcal{E}}(\nu, B_\lambda) \geq C_\lambda \tilde{\mathcal{E}}(\nu) \tag{3.8}$$

for all $\nu \in \mathcal{P}(\Gamma)$ and for all $\Gamma \ni 0$ such that $\operatorname{diam}(\Gamma) < \lambda/2$.

PROOF. If $x \in F_\lambda = \mathbb{E} \setminus B_\lambda$ and $y \in \Gamma$, then $|y| \leq \operatorname{diam}(\Gamma) < \lambda/2 \leq |x|/2$ and therefore $|x - y| > |x| - |y| \geq |x|/2$. This implies

$$\tilde{h}_\nu(x) \leq r(x) 2^d |x|^{-d}$$

and

$$\tilde{\mathcal{E}}(\nu, F_\lambda) \leq 2^{d\alpha} \int_{F_\lambda} r(x)^{\alpha+1} |x|^{-d\alpha} dx = C'_\lambda < \infty. \tag{3.9}$$

On the other hand, if $x \in B_\lambda, y \in \Gamma$, then $|x - y| \leq |x| + |y| \leq 3\lambda/2$. Therefore $\tilde{h}_\nu(x) \geq (3\lambda/2)^{-d} r(x)$ and

$$\tilde{\mathcal{E}}(\nu, B_\lambda) \geq (3\lambda/2)^{-d\alpha} \int_{B_\lambda} r(x)^{\alpha+1} dx = C''_\lambda > 0. \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$C'_\lambda \tilde{\mathcal{E}}(\nu, B_\lambda) \geq C'_\lambda C''_\lambda \geq C''_\lambda \tilde{\mathcal{E}}(\nu, F_\lambda) = C''_\lambda [\tilde{\mathcal{E}}(\nu) - \tilde{\mathcal{E}}(\nu, B_\lambda)]$$

and (3.8) holds with $C_\lambda = C''_\lambda / (C'_\lambda + C''_\lambda)$. □

3.3. Straightening of the boundary.

PROPOSITION 3.1. *Suppose that E is a bounded smooth domain. Then there exist strictly positive constants ε, a, b (depending only on E) such that, for every $x \in \partial E$:*

(a) The boundary can be straightened in $B(x, \varepsilon)$.

(b) The corresponding diffeomorphism ψ_x satisfies the conditions

$$a^{-1}|y_1 - y_2| \leq |\psi_x(y_1) - \psi_x(y_2)| \leq a|y_1 - y_2| \quad \text{for all } y_1, y_2 \in B(x, \varepsilon); \tag{3.11}$$

$$a^{-1} \operatorname{diam}(A) \leq \operatorname{diam}(\psi_x(A)) \leq a \operatorname{diam}(A) \quad \text{for all } A \subset B(x, \varepsilon); \tag{3.12}$$

$$a^{-1} d(A_1, A_2) \leq d(\psi_x(A_1), \psi_x(A_2)) \leq a\, d(A_1, A_2) \quad \text{for all } A_1, A_2 \subset B(x, \varepsilon). \tag{3.13}$$

$$b^{-1} \leq J_x(y) \leq b \quad \text{for all } y \in B(x, \varepsilon) \tag{3.14}$$

where $J_x(y)$ is the Jacobian of ψ_x at y.

Diffeomorphisms ψ_x can be chosen to satisfy additional conditions

$$\psi_x(x) = 0 \quad \text{and } \psi_x(B(x, \varepsilon)) \subset \mathbb{E}. \tag{3.15}$$

PROOF. The boundary ∂E can be covered by a finite number of balls $B_i = B(x_i, \varepsilon_i)$ in which straightening diffeomorphisms are defined. The function $q(x) = \max_i d(x, B_i^c)$ is continuous and strictly positive on ∂E. Therefore $\varepsilon = \frac{1}{2} \min_x q(x) > 0$. For every $x \in \partial E$ there exists B_i which contains the closure of $B(x, \varepsilon)$. We put

$$\psi_x(y) = \psi_{x_i}(y) \quad \text{for } y \in B(x, \varepsilon).$$

This is a diffeomorphism straightening ∂E in $B(x,\varepsilon)$.

For every x, $B(x,\varepsilon)$ is contained in one of closed balls $\tilde{B}_i = \{y : d(y, B_i^c) \geq \varepsilon\}$. Since ψ_{x_i} belongs to the class $C^{2,\lambda}(B_i)$, there exist constants $a_i > 0$ such that

$$a_i^{-1}|y_1 - y_2| \leq |\psi_{x_i}(y_1) - \psi_{x_i}(y_2)| \leq a_i|y_1 - y_2| \quad \text{for all } y_1, y_2 \in \tilde{B}_i.$$

The condition (3.11) holds for $a = \max a_i$. The conditions (3.12) and (3.13) follow from (3.11). The Jacobian J_{x_i} does not vanish at any point $y \in B_i$ and we can assume that it is strictly positive. The condition (3.14) holds because J_{x_i} is continuous on the closure of $B(x,\varepsilon)$.

By replacing $\psi_x(y)$ with $c[\psi_x(y) - \psi_x(x)]$ with a suitable constant c, we get diffeomorphisms subject to (3.15) in addition to (3.11)-(3.14). ☐

3.4. Proof of Theorem 2.1. 1°. If $\gamma < 0$, then (2.3) holds because $\mathrm{Cap}(\Gamma) \leq \mathrm{Cap}(\partial E) = C$. To prove (2.3) for $\gamma \geq 0$, it is sufficient to prove that, for some $\beta > 0$, there is a constant C_1 such that

$$\mathrm{Cap}(\Gamma) \leq C_1 \,\mathrm{diam}(\Gamma)^\gamma \quad \text{if } \mathrm{diam}(\Gamma) \leq \beta.$$

Indeed,

$$\mathrm{Cap}(\Gamma) \leq C_2 \,\mathrm{diam}(\Gamma)^\gamma \quad \text{if } \mathrm{diam}(\Gamma) \geq \beta$$

with $C_2 = \mathrm{Cap}(\partial E)\beta^{-\gamma}$.

2°. Let ε, a be the constants defined in Proposition 3.1 and let $\beta = \varepsilon/(2 + 8a^2) \wedge 1$. Suppose that $\mathrm{diam}(\Gamma) \leq \beta$ and let $x \in \Gamma$. Consider a straightening ψ_x of ∂E in $B(x,\varepsilon)$ which satisfies conditions (3.15). Put $B = B(x,\varepsilon), \tilde{B} = B(x,\varepsilon/2)$. By Lemma 3.2, there exists a smooth domain D such that $\tilde{B} \cap E \subset D \subset B \cap E$. Note that $\tilde{B} \cap \partial E \subset \partial D \cap \partial E \subset B \cap \partial E$. If $A_1 = \partial D \cap B \cap E$, then $d(x, A_1) \geq \varepsilon/2$ and $d(\Gamma, A_1) \geq \varepsilon/2 - \mathrm{diam}(\Gamma) \geq \varepsilon/2 - \beta$. Denote by D', Γ', A_1' the images of D, Γ, A_1 under ψ_x and let $A' = \mathbb{E} \backslash D'$. By (3.12), $\mathrm{diam}(\Gamma') \leq \lambda_1 = a\beta$ and $d(\Gamma', A') \geq \lambda_2 = (\varepsilon/2 - \beta)/a$. Our choice of β implies that $\lambda_1 < \lambda_2/4$. Put $\lambda = \lambda_1 + \lambda_2/4$. Note that $\lambda_2 > 2\lambda$ and $\lambda_1 < \lambda/2$. Since $d(\Gamma', A') = d(\Gamma, A_1')$, Lemmas 3.3 and 3.4 are applicable to D', Γ', A' and λ (which depends only on E).

3°. By **2**.(1.10) and (3.13), for every $y \in D, z \in \Gamma$,

$$k_E(y,z) \geq Cd(y, \partial E)|y - z|^{-d} \geq Cd(y, \partial D)|y - z|^{-d} \geq Cd(y', \partial D')|y' - z'|^{-d} \tag{3.16}$$

where $y' = \psi_x(y), z' = \psi_x(z)$. If ν' is the image of $\nu \in \mathcal{P}(\Gamma)$ under ψ_x, then

$$\int_\Gamma f[\psi_x(z)]\nu(dz) = \int_{\Gamma'} f(z')\nu'(dz')$$

for every positive measurable function f. In particular,

$$\int_\Gamma |y' - \psi_x(z)|^{-d}\nu(dz) = \int_{\Gamma'} |y' - z'|^{-d}\nu'(dz'). \tag{3.17}$$

By (3.16) and (3.17),

$$\int_\Gamma k_E(y,z)\nu(dz) \ge Cd(y',\partial D')\int_{\Gamma'}|y'-z'|^{-d}\nu'(dz').$$

If $y' \in B_\lambda$, then, by Lemma 3.3, $d(y',\partial D') = r(y')$ and we have

$$h_\nu(y) = \int_\Gamma k_E(y,z)\nu(dz) \ge C\int_{\Gamma'} r(y')|y'-z'|^{-d}\nu'(dz') = C\tilde{h}_{\nu'}[\psi_x(y)]. \tag{3.18}$$

If $y \in D$, then, by (3.13), $d(y,\partial E) \ge d(y,\partial D) \ge Cd(y',\partial D')$ and therefore (1.1), (2.1) and (3.18) imply

$$\begin{aligned}\mathcal{E}(\nu) = \int_E d(y,\partial E)h_\nu(y)^\alpha dy &\ge \int_D d(y,\partial D)h_\nu(y)^\alpha dy \\ &\ge C\int_D d(\psi_x(y),\partial D')\tilde{h}_{\nu'}[\psi_x(y)]^\alpha dy.\end{aligned} \tag{3.19}$$

Note that

$$\int_{D'} f(y')dy' = \int_D f[\psi_x(y)]J_x(y)dy$$

and, if $f \ge 0$, then, by (3.14),

$$\int_{D'} f(y')dy' \le b\int_D f[\psi_x(y)]dy.$$

By taking $f(y') = d(y',\partial D')\tilde{h}_{\nu'}(y')^\alpha$, we get from (3.19)

$$\mathcal{E}(\nu) \ge C\int_{D'} d(y',\partial D')\tilde{h}_{\nu'}(y')^\alpha dy'.$$

By Lemma 3.3, $D' \supset B_\lambda$ and $d(y',\partial D') = r(y')$ on B_λ. Hence

$$\mathcal{E}(\nu) \ge C\int_{B_\lambda} r(y')\tilde{h}_{\nu'}(y')^\alpha dy' = C\tilde{\mathcal{E}}(\nu',B_\lambda).$$

By Lemma 3.4, we have $\mathcal{E}(\nu) \ge C\tilde{\mathcal{E}}(\nu')$ and by (1.3), $\mathrm{Cap}(\Gamma) \le C\widehat{\mathrm{Cap}}(\Gamma')$. The bound $\mathrm{Cap}(\Gamma) \le C\,\mathrm{diam}(\Gamma)^\gamma$ follows from Lemma 3.1, (3.12) and 1°. □

4. Lower bound for Cap_x

4.1. Put

$$\begin{gathered}\delta(x) = d(x,\Gamma), \quad E_1 = \{x \in E : \delta(x) < 3\rho(x)/2\}, \quad E_2 = E \setminus E_1; \\ \mathcal{E}_x(\nu,B) = \int_B g(x,y)h_\nu(y)^\alpha dy \quad \text{for } B \subset E\end{gathered} \tag{4.1}$$

and let

$$U_x = \{y \in E : |x-y| < \delta(x)/2\}, \qquad V_x = \{y \in E : |x-y| \ge \delta(x)/2\}. \tag{4.2}$$

First, we deduce Theorem 2.2 from the following three lemmas. Then we prove these lemmas.

LEMMA 4.1. *For all* Γ, *all* $\nu \in \mathcal{P}(\Gamma)$ *and all* $x \in E$,

$$\mathcal{E}_x(\nu, V_x) \leq C\varphi(x,\Gamma)\mathcal{E}(\nu). \tag{4.3}$$

LEMMA 4.2. *For all* Γ, *all* $\nu \in \mathcal{P}(\Gamma)$ *and all* $x \in E_1$,

$$\mathcal{E}_x(\nu, U_x) \leq C\varphi(x,\Gamma)\mathcal{E}(\nu). \tag{4.4}$$

LEMMA 4.3. *For all* Γ, *all* $\nu \in \mathcal{P}(\Gamma)$ *and all* $x \in E_2$,

$$\mathcal{E}_x(\nu, U_x) \leq C\varphi(x,\Gamma)\theta(x)^{-\gamma_+}\mathcal{E}(\nu) \tag{4.5}$$

where

$$\theta(x) = d(x,\Gamma)/\operatorname{diam}(\Gamma).$$

4.2. Proof of Theorem 2.2. By Lemmas 4.2 and 4.3, for every $x \in E$

$$\mathcal{E}_x(\nu, U_x) \leq C\varphi(x,\Gamma)\mathcal{E}(\nu)(1 \vee \theta(x)^{-\gamma_+})$$

and therefore, under the condition (2.8),

$$\mathcal{E}_x(\nu, U_x) \leq C\varphi(x,\Gamma)\mathcal{E}(\nu)(1 \vee \kappa^{-\gamma_+}).$$

This bound and Lemma 4.1 imply that

$$\mathcal{E}_x(\nu) = \mathcal{E}_x(\nu, U_x) + \mathcal{E}_x(\nu, V_x) \leq C\varphi(x,\Gamma)\mathcal{E}(\nu)[2 \vee (1+\kappa^{-\gamma_+})]$$

and, by (1.3),

$$\operatorname{Cap}_x(\Gamma) \geq C_\kappa \varphi(x,\Gamma)^{-1}\operatorname{Cap}(\Gamma) \tag{4.6}$$

where $C_\kappa = C^{-1}[2 \vee (1+\kappa^{-\gamma_+})]^{-1}$. If α is subcritical, then $\gamma < 0$, C_κ does not depend on κ and (4.6) implies (2.6). If α is supercritical, then $\gamma \geq 0$ and (2.7) holds under the condition (2.8). $\square$

4.3. Proof of Lemma 4.1. By **2**.(1.7),

$$\mathcal{E}_x(\nu, V_x) \leq C\rho(x)\int_{V_x}\rho(y)|x-y|^{-d}h_\nu(y)^\alpha dy.$$

Since $|x-y| \geq \delta(x)/2$ for $y \in V_x$, this implies (4.3). $\square$

4.4. Proof of Lemma 4.2. The function h_ν is harmonic in the ball $\{y : |x-y|/\rho(x) \leq r$ for $0 < r < 1$. By the Harnack's inequality,

$$\frac{1-r}{(1+r)^{d-1}}h_\nu(x) \leq h_\nu(y) \leq \frac{1+r}{(1-r)^{d-1}}h_\nu(x) \tag{4.7}$$

(see, e.g. [GT98], p.29, Problem 2.6). If $x \in E_1, y \in U_x$, then $|x-y| < \delta(x)/2 < 3\rho(x)/4$ and (4.7) holds with $r = 3/4$. Therefore, for all $x \in E_1, y \in U_x$, $C'_d h_\nu(x) \leq h_\nu(y) \leq C''_d h_\nu(x)$ where C'_d and C''_d depend only on d. This implies bounds

$$\mathcal{E}_x(\nu, U_x) \leq C''_d h_\nu(x)^\alpha \int_{U_x} g_E(x,y)dy \tag{4.8}$$

and

$$\mathcal{E}(\nu) \geq \int_{U_x}\rho(y)h_\nu(y)^\alpha dy \geq C'_d h_\nu(x)^\alpha \int_{U_x}\rho(y)dy. \tag{4.9}$$

By **2**.(1.6),
(4.10)
$$\int_{U_x} g_E(x,y)dy \le C\rho(x)\int_{U_x}|x-y|^{1-d}dy = C\rho(x)\int_0^{\delta(x)/2} dt \le C\delta(x)\rho(x).$$
For $y \in U_x, x \in E_1$,
$$\rho(y) \ge \rho(x) - |x-y| \ge \rho(x) - \delta(x)/2 \ge \rho(x)/4$$
and therefore
$$\int_{U_x}\rho(y)dy \ge \frac{1}{4}\rho(x)\int_{U_x} dy = C_d\rho(x)\delta(x)^d. \tag{4.11}$$
Since $\delta\rho \le 3\varphi\rho\delta^d/2$, bound (4.4) follows from (4.8)–(4.11). □

4.5. Proof of Lemma 4.3. By Theorem 2.1,
$$\mathcal{E}(\nu)^{-1} \le \mathrm{Cap}(\Gamma) \le C\,\mathrm{diam}(\Gamma)^{\gamma_+}.$$
Hence,
$$\mathrm{diam}(\Gamma)^{-\gamma_+} \le C\mathcal{E}(\nu). \tag{4.12}$$
If $x \in E_2$ and $y \in U_x$, then $\delta(y) \ge \delta(x) - |x-y| > \delta(x)/2$ and $\rho(y) \le \rho(x) + |x-y| \le 2\delta(x)/3 + \delta(x)/2 = 7\delta(x)/6$. For all $z \in \Gamma, y \in U_x$, $|y-z| \ge |z-x| - |y-x| \ge \delta(x)/2$ and, by **2**.(1.10),
$$k_E(y,z) \le C\rho(y)|y-z|^{-d} \le C\delta(x)^{1-d}.$$
Therefore $h_\nu(y) \le C\delta(x)^{1-d}$ and, by **2**.(1.6),
$$\mathcal{E}_x(\nu, U_x) \le C\rho(x)\delta(x)^{(1-d)\alpha}\int_{U_x}|x-y|^{1-d}dy \le C\varphi(x,\Gamma)\delta(x)^{-\gamma}. \tag{4.13}$$
If $\gamma < 0$, then $\delta(x)^{-\gamma} \le \mathrm{diam}(E)^{-\gamma} = C$. If $\gamma \ge 0$, then $\gamma = \gamma_+$. Hence, the bound (4.5) follows from (4.12) and (4.13). □

5. Notes

The capacity Cap defined by the formulae (1.3)–(1.5) with m defined by (2.1) is related to a Poisson capacity CP_α used in **[D]** by the equation
$$\mathrm{Cap}(\Gamma) = \mathrm{CP}_\alpha(\Gamma)^\alpha.$$
[The capacity CP_α is a particular case of the Martin capacity also considered in **[D]**. The Martin kernel is a continuous function on $E \times \tilde{E}$ where E is a domain on $\mathbb{R}^d$ (not necessarily smooth) and $\tilde{E}$ is the Martin boundary of E for an L-diffusion.]

Let Cap^L and Cap_x be the Poisson capacities corresponding to an operator L. It follows from **2**.(1.10) that, for every L_1 and L_2, the ratio $\mathrm{Cap}^{L_1}/\mathrm{Cap}^{L_2}$ is bounded and therefore we can restrict ourselves by the Poisson capacities corresponding to the Laplacian Δ.

The capacity CP_α was introduced in [DK96b] as a tool for a study of removable boundary singularities for solutions of the equation $Lu = u^\alpha$. It was proved that, if E is a bounded smooth domain, then a closed subset Γ of ∂E is a removable singularity if and only if $\mathrm{CP}_\alpha(\Gamma) = 0$. First, this was conjectured in [Dy94]. In the case $\alpha = 2$, the conjecture was proved by Le Gall [Le95] who used the capacity Cap^∂. Le Gall's capacity Cap^∂ has the same class of null sets as CP_α.

An analog of formula (2.7) with Cap replaced by Cap^∂ follows from formula (3.34) in [Ms04] in the case $L = \Delta, \alpha = 2, d \geq 4$ and $\kappa = 4$.

The results presented in Chapter 6 were published, first, in [DK03].

CHAPTER 7

Basic inequality

In this chapter we consider two smooth domains $D \subset E$, the set

$$D^* = \{x \in \bar{D} : d(x, E \setminus D) > 0\} \tag{0.1}$$

and measures ν concentrated on $\partial D \cap \partial E$. Our goal is to give a lower bound of $\mathbb{N}_x\{\mathcal{R}_E \subset D^*, Z_\nu \neq 0\}$ in terms of $\mathbb{N}_x\{\mathcal{R}_E \subset D^*, Z_\nu\}$ and $\mathcal{E}_x(\nu)$. This bound will play an important role in proving the equation $u_\Gamma = w_\Gamma$ in Chapter **8**.

Preparations for proving the basic inequality include: (a) establishing relations between $\mathcal{R}_E$ and $\mathcal{R}_D$ and between stochastic boundary values in E and D; (b) expressing certain integrals with respect to the measures P_x and $\mathbb{N}_x$ through the conditional diffusion Π_x^ν.

1. Main result

THEOREM 1.1. *Suppose that D is a smooth open subset of a smooth domain E. If ν is a finite measure concentrated on $\partial D \cap \partial E$ and if $\mathcal{E}_x(\nu) < \infty$, then*

(1.1)

$$\mathbb{N}_x\{\mathcal{R}_E \subset D^*, Z_\nu \neq 0\} \geq C(\alpha)[\mathbb{N}_x\{\mathcal{R}_E \subset D^*, Z_\nu\}]^{\alpha/(\alpha-1)}\mathcal{E}_x(\nu)^{-1/(\alpha-1)}$$

where $C(\alpha) = (\alpha - 1)^{-1}\Gamma(\alpha - 1)$. [1]

REMARK. By **3**.3.4.C, the condition $\mathcal{E}_x(\nu) < \infty$ implies that ν belongs to $\mathcal{N}_1^E$ and to $\mathcal{N}_1^D$.

2. Two propositions

2.1.

PROPOSITION 2.1. *Suppose $x \in D$, Λ is a Borel subset of ∂D and $\mathcal{A} = \{\mathcal{R}_D \cap \Lambda = \emptyset\}$. We have $P_x\mathcal{A} > 0$ and for all $Z', Z'' \in \mathcal{Z}_x$,*

$$\begin{aligned} &\mathbb{N}_x\{\mathcal{A}, (e^{-Z'} - e^{-Z''})^2\} \\ &\quad = -2\log P_x\{e^{-Z'-Z''} \mid \mathcal{A}\} + \log P_x\{e^{-2Z'} \mid \mathcal{A}\} + \log P_x\{e^{-2Z''} \mid \mathcal{A}\}. \end{aligned} \tag{2.1}$$

If $Z' = Z''$ P_x-a.s. on $\mathcal{A}$ and if $P_x\{\mathcal{A}, Z' < \infty\} > 0$, then $Z' = Z''$ $\mathbb{N}_x$-a.s. on $\mathcal{A}$.

[1]Here Γ is Euler's Gamma-function.

PROOF. First, $P_x\mathcal{A} > 0$ because, by **3**.(3.9), $P_x\mathcal{A} = e^{-w_\Lambda(x)}$. Next,

$$(e^{-Z'} - e^{-Z''})^2 = 2(1 - e^{-Z'-Z''}) - (1 - e^{-2Z'}) - (1 - e^{-2Z''}).$$

Therefore (2.1) follows from **4**.(3.28). The second part of the proposition is an obvious implication of (2.1). □

2.2. Note that

$$D^* = \{x \in \bar{D} : d(x, \Lambda) > 0\} \tag{2.2}$$

where $\Lambda = \partial D \cap E$.

PROPOSITION 2.2. *Let $D \subset E$ be two open sets. Then, for every $x \in D$, X_D and X_E coincide P_x-a.s. and $\mathbb{N}_x$-a.s. on the set $\mathcal{A} = \{\mathcal{R}_D \subset D^*\}$.*

PROOF. By the Markov property **3**.2.1.D, for every Borel set B,

$$P_x\{\mathcal{A}, e^{-X_E(B)}\} = P_x\{\mathcal{A}, P_{X_D} e^{-X_E(B)}\}. \tag{2.3}$$

Suppose $x \in D$. Then $X_D(D) = 0$ P_x-a.s. by **3**.2.2.A, and $X_D(\partial D \cap E) = 0$ P_x-a.s. on $\mathcal{A}$ because X_D is concentrated on $\mathcal{R}_D$. Hence, P_x-a.s., $X_D(E) = 0$ on $\mathcal{A}$ and, by **3**.2.1.C,

$$P_{X_D} e^{-X_E(B)} = e^{-X_D(B)}. \tag{2.4}$$

By (2.3) and (2.4)

$$P_x\{\mathcal{A}, e^{-X_E(B)}\} = P_x\{\mathcal{A}, e^{-X_D(B)}\}. \tag{2.5}$$

Put $C_1 = \partial D \cap \partial E, C_0 = \partial E \setminus C_1$. By **3**.2.2.A, $P_x\{X_D(C_0) = 0\} = 1$ and (2.5) implies that $X_E(C_0) = 0$ P_x-a.s. on $\mathcal{A}$. On the other hand, if $B \subset C_1$, then $P_x\{X_D(B) \le X_E(B)\} = 1$ by **3**.2.1.E and therefore $X_D(B) = X_E(B)$ P_x-a.s. on $\mathcal{A}$. We conclude that $X_D = X_E$ P_x-a.s. on $\mathcal{A}$.

Now we apply Proposition 2.1 to $Z' = X_D(B), Z'' = X_E(B)$ and $\Lambda = \partial D \cap E$. Note that, by **3**.2.2.B, $P_x Z' = K_D(x, B) < \infty$. Therefore $P_x\{\mathcal{A}, Z'\} < \infty$ and $P_x\{\mathcal{A}, Z' < \infty\} > 0$. By Proposition 2.1, $Z' = Z''$ $\mathbb{N}_x$-a.s. on $\mathcal{A}$. □

3. Relations between superdiffusions and conditional diffusions in two open sets

3.1. Now we consider two bounded smooth open sets $D \subset E$. We denote by $\tilde{Z}_\nu$ the stochastic boundary value of $\tilde{h}_\nu(x) = \int_{\partial D} k_D(x, y)\nu(dy)$ in D; $\tilde{\Pi}_x^y$ refers to the diffusion in D conditioned to exit at $y \in \partial D$.

THEOREM 3.1. *Put $\mathcal{A} = \{\mathcal{R}_D \subset D^*\}$. For every $x \in D$,*

$$\mathcal{R}_E = \mathcal{R}_D \quad P_x\text{-a.s. and} \quad \mathbb{N}_x\text{-a.s. on} \mathcal{A} \tag{3.1}$$

and

$$Z_\nu = \tilde{Z}_\nu \quad P_x\text{-a.s. and} \quad \mathbb{N}_x\text{-a.s. on } \mathcal{A} \tag{3.2}$$

for all $\nu \in \mathcal{N}_1^E$ concentrated on $\partial D \cap \partial E$.

PROOF. 1°. First, we prove (3.1). Clearly, $\mathcal{R}_D \subset \mathcal{R}_E$ P_x-a.s. and $\mathbb{N}_x$-a.s. for all $x \in D$. We get (3.1) if we show that, if O is an open subset of E, then, for every $x \in D$, $X_O = X_{O\cap D}$ P_x-a.s. on $\mathcal{A}$ and, for every $x \in O \cap D$, $X_O = X_{O\cap D}$ $\mathbb{N}_x$-a.s. on $\mathcal{A}$. For $x \in O \cap D$ this follows from Proposition 2.2 applied to $O \cap D \subset O$ because $\{\mathcal{R}_D \subset D^*\} \subset \{\mathcal{R}_{O\cap D} \subset (O \cap D)^*\}$. For $x \in D \setminus O$, $P_x\{X_O = X_{D\cap O} = \delta_x\} = 1$ by **3**.2.1.C.

2°. Put

$$D^*_m = \{x \in \bar{D} : d(x, E \setminus D) > 1/m\}. \tag{3.3}$$

To prove (3.2), it is sufficient to prove that it holds on $\mathcal{A}_m = \{\mathcal{R}_D \subset D^*_m\}$ for all sufficiently large m. First we prove that, for all $x \in D$,

$$Z_\nu = \tilde{Z}_\nu \quad P_x\text{-a.s. on } \mathcal{A}_m. \tag{3.4}$$

We get (3.4) by proving that both Z_ν and $\tilde{Z}_\nu$ coincide P_x-a.s. on $\mathcal{A}_m$ with the stochastic boundary value Z^* of h_ν in D.

Let

$$E_n = \{x \in E : d(x, \partial E) > 1/n\}, \quad D_n = \{x \in D : d(x, \partial D) > 1/n\}.$$

If $n > m$, then

$$\mathcal{A}_m \subset \mathcal{A}_n \subset \{\mathcal{R}_D \subset D^*_n\} \subset \{\mathcal{R}_{D_n} \subset D^*_n\}.$$

We apply Proposition 2.2 to $D_n \subset E_n$ and we get that, P_x-a.s. on $\{\mathcal{R}_{D_n} \subset D^*_n\} \supset \mathcal{A}_m$, $X_{D_n} = X_{E_n}$ for all $n > m$ which implies $Z^* = Z_\nu$.

3°. Now we prove that

$$Z^* = \tilde{Z}_\nu \quad P_x\text{-a.s. on } \mathcal{A}_m. \tag{3.5}$$

Consider $h^0 = h_\nu - \tilde{h}_\nu$ and $Z^0 = Z^*_\nu - \tilde{Z}_\nu$. By **3**.1.1.C, if $y \in \partial D \cap \partial E$, then

$$k_E(x, y) = k_D(x, y) + \Pi_x\{\tau_D < \tau_E, k_E(\xi_{\tau_D}, y)\}. \tag{3.6}$$

Therefore

$$h^0(x) = \Pi_x\{\xi_{\tau_D} \in \partial D \cap E, h_\nu(\xi_{\tau_D})\}. \tag{3.7}$$

This is a harmonic function in D. By **2**.2.3.C, it vanishes on $\Gamma_m = \partial D \cap D^*_m = \partial E \cap D^*_m$.

We claim that, for every $\varepsilon > 0$ and every m, $h^0 < \varepsilon$ on $\Gamma_{m,n} = \partial E_n \cap D^*_m$ for all sufficiently large n. [If this is not true, then there exists a sequence $n_i \to \infty$ such that $z_{n_i} \in \Gamma_{m,n_i}$ and $h^0(z_{n_i}) \geq \varepsilon$. If z is limit point of z_{n_i}, then $z \in \Gamma_m$ and $h^0(z) \geq \varepsilon$.]

All measures X_{D_n} are concentrated, P_x-a.s., on $\mathcal{R}_D$. Therefore $\mathcal{A}_m$ implies that they are concentrated, P_x-a.s., on D^*_m. Since $\Gamma_{m,n} \subset D^*_m$, we conclude that, for all sufficiently large n, $\langle h^0, X_{D_n}\rangle < \varepsilon\langle 1, X_{D_n}\rangle$ P_x-a.s. on $\mathcal{A}_m$. This implies (3.5).

4°. If $\nu \in \mathcal{M}(\partial E)$ and $Z_\nu = \mathrm{SBV}(h_\nu)$, then, by **3**.3.6.A and Remark **4**.3.1,

$$\mathbb{N}_x Z_\nu = P_x Z_\nu \leq h_\nu(x) < \infty. \tag{3.8}$$

Note that $P_x\mathcal{A} > 0$. It follows from (3.8) that $Z_\nu < \infty$ P_x-a.s. and therefore $P_x\{\mathcal{A}, Z_\nu < \infty\} > 0$. By Proposition 2.1, (3.2) follows from (3.4) .

□

4. Equations connecting P_x and $\mathbb{N}_x$ with Π_x^ν

4.1.

Theorem 4.1. *Let* $Z_\nu = \mathrm{SBV}(h_\nu), Z_u = \mathrm{SBV}(u)$ *where* $\nu \in \mathcal{N}_1^E$ *and* $u \in \mathcal{U}(E)$. *Then*

$$P_x Z_\nu e^{-Z_u} = e^{-u(x)} \Pi_x^\nu e^{-\Phi(u)} \tag{4.1}$$

and

$$\mathbb{N}_x Z_\nu e^{-Z_u} = \Pi_x^\nu e^{-\Phi(u)} \tag{4.2}$$

where

$$\Phi(u) = \int_0^{\tau_E} \psi'[u(\xi_t)]dt \tag{4.3}$$

Proof. Formula (4.1) follows from **[D]**, Theorem 9.3.1. To prove (4.2), we observe that, for every $\lambda > 0$, $h_{\lambda\nu} + u \in \mathcal{U}^-$ by **2**.3.D, and therefore

$$\mathbb{N}_x(1 - e^{-\lambda Z_\nu - Z_u}) = -\log P_x e^{-\lambda Z_\nu - Z_u} \tag{4.4}$$

by Theorem **4**.3.2. By taking the derivatives with respect to λ at $\lambda = 0$, [2] we get

$$\mathbb{N}_x Z_\nu e^{-Z_u} = P_x Z_\nu e^{-Z_u} / P_x e^{-Z_u}.$$

By **3**.(3.4), $P_x e^{-Z_u} = e^{-u(x)}$ and therefore (4.2) follows from (4.1). □

Theorem 4.2. *Suppose that* $D \subset E$ *are bounded smooth open sets and* $\Lambda = \partial D \cap E$. *Let* ν *be a finite measure on* $\partial D \cap \partial E$, $x \in E$ *and* $\mathcal{E}_x(\nu) < \infty$. *Put*

$$\begin{aligned} w_\Lambda(x) &= \mathbb{N}_x\{\mathcal{R}_D \cap \Lambda \neq \emptyset\}, \\ v_s(x) &= w_\Lambda(x) + \mathbb{N}_x\{\mathcal{R}_D \cap \Lambda = \emptyset, 1 - e^{-sZ_\nu}\} \end{aligned} \tag{4.5}$$

for $x \in D$ *and let* $w_\Lambda(x) = v_s(x) = 0$ *for* $x \in E \setminus D$. *For every* $x \in E$, *we have*

$$\mathbb{N}_x\{\mathcal{R}_E \subset D^*, Z_\nu\} = \Pi_x^\nu\{A, e^{-\Phi(w_\Lambda)}\}, \tag{4.6}$$

$$\mathbb{N}_x\{\mathcal{R}_E \subset D^*, Z_\nu \neq 0\} = \int_0^\infty \Pi_x^\nu\{A, e^{-\Phi(v_s)}\} ds \tag{4.7}$$

where Φ *is defined by* (4.3) *and*

$$A = \{\tau_E = \tau_D\} = \{\xi_t \in D \quad \textit{for all } t < \tau_E\}. \tag{4.8}$$

[2]The differentiation under the integral signs is justified by **4**.(3.8). [In the setting of a Brownian snake formula (4.2) can be found in [Ms04] (see Proposition 2.31).]

PROOF. 1°. If $x \in E \setminus D$, then, $\mathbb{N}_x$-a.s., $\mathcal{R}_E$ is not a subset D^*. Indeed, $\mathcal{R}_E$ contains supports of X_O for all neighborhoods O of x and therefore $x \in \mathcal{R}_E$ P_x-a.s. Hence, $\mathbb{N}_x\{\mathcal{R}_E \subset D^*\} = 0$. On the other hand, $\Pi_x^\nu(A) = 0$. Therefore (4.6) and (4.7) hold independently of values of w_Λ and v_s.

2°. Now we assume that $x \in D$. Put $\mathcal{A} = \{\mathcal{R}_D \subset D^*\}$. We claim that

$$\mathcal{A} = \{\mathcal{R}_E \subset D^*\} \quad \mathbb{N}_x\text{-a.s.}$$

Indeed, $\{\mathcal{R}_E \subset D^*\} \subset \mathcal{A}$ because $\mathcal{R}_D \subset \mathcal{R}_E$. By Theorem 3.1, $\mathbb{N}_x$-a.s., $\mathcal{A} \subset \{\mathcal{R}_D = \mathcal{R}_E\}$ and therefore $\mathcal{A} \subset \{\mathcal{R}_E \subset D^*\}$.

By Theorem 3.1, $\mathcal{R}_D = \mathcal{R}_E$ and $Z_\nu = \tilde{Z}_\nu$ $\mathbb{N}_x$-a.s. on $\mathcal{A}$. Therefore

$$\begin{aligned} &\mathbb{N}_x\{\mathcal{R}_E \subset D^*, Z_\nu\} = \mathbb{N}_x\{\mathcal{A}, Z_\nu\} = \mathbb{N}_x\{\mathcal{A}, \tilde{Z}_\nu\}, \\ &\mathbb{N}_x\{\mathcal{R}_E \subset D^*, Z_\nu e^{-sZ_\nu}\} = \mathbb{N}_x\{\mathcal{A}, Z_\nu e^{-sZ_\nu}\} = \mathbb{N}_x\{\mathcal{A}, \tilde{Z}_\nu e^{-s\tilde{Z}_\nu}\}. \end{aligned} \tag{4.9}$$

By Theorem **4**.3.4, $v_s = w_\Lambda \oplus u_{s\nu}$. Let Z_Λ, Z^s and $\tilde{Z}_{s\nu}$ be the stochastic boundary values in D of w_Λ, v_s and $u_{s\nu}$. By **3**.3.5.A, $Z_\Lambda = \infty \cdot 1_{\mathcal{A}^c}$ and therefore

$$e^{-Z_\Lambda} = 1_{\mathcal{A}}. \tag{4.10}$$

By **3**.3.3.B, $Z^s = Z_\Lambda + \tilde{Z}_{s\nu}$. Hence,

$$e^{-Z^s} = 1_{\mathcal{A}} e^{-s\tilde{Z}_\nu}. \tag{4.11}$$

By (4.9), (4.10) and (4.11),

$$\mathbb{N}_x\{\mathcal{A}, Z_\nu\} = \mathbb{N}_x\{1_{\mathcal{A}}\tilde{Z}_\nu\} = \mathbb{N}_x\{\tilde{Z}_\nu e^{-Z_\Lambda}\} \tag{4.12}$$

and

$$\mathbb{N}_x\{\mathcal{A}, Z_\nu e^{-sZ_\nu}\} = \mathbb{N}_x\{1_{\mathcal{A}}\tilde{Z}_\nu e^{-s\tilde{Z}_\nu}\} = \mathbb{N}_x\{\tilde{Z}_\nu e^{-Z^s}\}. \tag{4.13}$$

By applying formula (4.2) to $\tilde{Z}_\nu$ and the restriction of w_Λ to D, we conclude from (4.12) that

$$\mathbb{N}_x\{\mathcal{A}, Z_\nu\} = \tilde{\Pi}_x^\nu \exp\big[-\int_0^{\tau_D} \psi'[w_\Lambda(\xi_s)]ds\big] \tag{4.14}$$

and, by **3**.(1.16),

$$\mathbb{N}_x\{\mathcal{A}, Z_\nu\} = \Pi_x^\nu\{A, e^{-\Phi(w_\Lambda)}\}. \tag{4.15}$$

Analogously, by applying (4.2) to $\tilde{Z}_\nu$ and the restriction of v_s to D, we get from (4.13) and **3**.(1.16) that

$$\mathbb{N}_x\{\mathcal{A}, Z_\nu e^{-sZ_\nu}\} = \Pi_x^\nu\{A, e^{-\Phi(v_s)}\}. \tag{4.16}$$

Formula (4.6) follows from (4.15) and formula (4.7) follows from (4.16) because

$$\mathbb{N}_x\{\mathcal{A}, Z_\nu \neq 0\} = \lim_{t\to\infty} \mathbb{N}_x\{\mathcal{A}, 1 - e^{-tZ_\nu}\} \tag{4.17}$$

and

$$1 - e^{-tZ_\nu} = \int_0^t Z_\nu e^{-sZ_\nu} ds. \tag{4.18}$$

□

5. Proof of Theorem 1.1

We use the following two elementary inequalities:

5.A. For all $a, b \geq 0$ and $0 < \beta < 1$,

$$(a+b)^\beta \leq a^\beta + b^\beta. \tag{5.1}$$

Proof. It is sufficient to prove (5.1) for $a = 1$. Put $f(t) = (1+t)^\beta - t^\beta$. Note that $f(0) = 1$ and $f'(t) \leq 0$ for $t > 0$. Hence $f(t) \leq 1$ for $t \geq 0$. □

5.B. For every finite measure M, every positive measurable function Y and every $\beta > 0$,

$$M(Y^{-\beta}) \geq M(1)^{1+\beta}(MY)^{-\beta}.$$

Indeed $f(y) = y^{-\beta}$ is a convex function on $\mathbb{R}_+$, and we get 5.B by applying Jensen's inequality to the probability measure $M/M(1)$.

Proof of Theorem 1.1. 1°. If $x \in E \setminus D$, then, $\mathbb{N}_x$-a.s., $\mathcal{R}_E$ is not a subset D^* (see proof of Theorem 4.2). Hence, both parts of (1.1) vanish.

2°. Suppose $x \in D$. Since $\nu \in \mathcal{N}_1^E$, it follows from Theorem **4**.3.4 that $\mathbb{N}_x(1 - e^{-sZ_\nu}) = u_{s\nu}(x)$. Thus (4.5) implies $v_s \leq w_\Lambda + u_{s\nu}$. Therefore, by 5.A, $v_s^{\alpha-1} \leq w_\Lambda^{\alpha-1} + u_{s\nu}^{\alpha-1}$ and, since $u_{s\nu} \leq h_{s\nu} = sh_\nu$, $\Phi(v_s) \leq \Phi(w_\Lambda) + s^{\alpha-1}\Phi(h_\nu)$.

Put $\mathcal{A} = \{\mathcal{R}_E \subset D^*\}$. It follows from (4.7) that

$$\mathbb{N}_x\{\mathcal{A}, Z_\nu \neq 0\} \geq \Pi_x^\nu\{A, \int_0^\infty e^{-\Phi(w_\Lambda) - s^{\alpha-1}\Phi(h_\nu)} ds\}. \tag{5.2}$$

Note that $\int_0^\infty e^{-as^\beta} ds = Ca^{-1/\beta}$ where $C = \int_0^\infty e^{-t^\beta} dt$. Therefore (5.2) implies

$$\mathbb{N}_x\{\mathcal{A}, Z_\nu \neq 0\} \geq C\Pi_x^\nu\{A, e^{-\Phi(w_\Lambda)}\Phi(h_\nu)^{-1/(\alpha-1)}\} = CM(Y^{-\beta}) \tag{5.3}$$

where $\beta = 1/(\alpha - 1), Y = \Phi(h_\nu)$ and M is the measure with the density $1_A e^{-\Phi(w_\Lambda)}$ with respect to Π_x^ν. We get from (5.3) and 5.B, that

$$\begin{aligned}\mathbb{N}_x\{\mathcal{A}, Z_\nu \neq 0\} &\geq CM(1)^{1+\beta}(MY)^{-\beta} \\ &= C[\Pi_x^\nu\{A, e^{-\Phi(w_\Lambda)}\}]^{\alpha/(\alpha-1)}[\Pi_x^\nu\{A, e^{-\Phi(w_\Lambda)}\Phi(h_\nu)\}]^{-1/(\alpha-1)}.\end{aligned}$$

By (4.6), $\Pi_x^\nu\{A, e^{-\Phi(w_\Lambda)}\} = \mathbb{N}_x\{\mathcal{R}_E \subset D^*, Z_\nu\}$ and since

$$\Pi_x^\nu\{A, e^{-\Phi(w_\Lambda)}\Phi(h_\nu)\} \leq \Pi_x^\nu\Phi(h_\nu),$$

we have

$$\mathbb{N}_x\{\mathcal{A}, Z_\nu \neq 0\} \geq C[\mathbb{N}_x\{\mathcal{R}_E \subset D^*, Z_\nu\}]^{\alpha/(\alpha-1)}[\Pi_x^\nu\Phi(h_\nu)]^{-1/(\alpha-1)}. \tag{5.4}$$

3°. By **3**.1.3.A, for every $f \in \mathcal{B}(E)$ and every $h \in \mathcal{H}(E)$,

$$\Pi_x^h \int_0^{\tau_E} f(\xi_t)dt = \int_0^\infty \Pi_x^h\{t < \tau_E, f(\xi_t)\}dt = \int_0^\infty \Pi_x\{t < \tau_E, f(\xi_t)h(\xi_t)\}dt.$$

By taking $f = \alpha h_\nu^{\alpha-1}$ and $h = h_\nu$ we get

$$\Pi_x^\nu \Phi(h_\nu) = \alpha \mathcal{E}_x(\nu). \tag{5.5}$$

Formula (1.1) follows from (5.4) and (5.5). □

6. Notes

The role of the basic inequality (1.1) in the investigation of the equation $Lu = u^\alpha$ is similar to the role of the formula (3.31) in Mselati's paper [Ms04]. In our notation, his formula can be written as

$$\mathbb{N}_x\{\mathcal{R}_E \cap \Lambda = \emptyset, Z_\nu \neq 0\} \geq [\mathbb{N}_x\{\mathcal{R}_E \cap \Lambda = \emptyset, Z_\nu\}]^2 [\mathbb{N}_x(Z_\nu^2)]^{-1} \tag{6.1}$$

which follows at once from the Cauchy-Schwarz inequality. A natural idea to write an analog of (6.1) by using the Hölder inequality does not work because $\mathbb{N}_x(Z_\nu^\alpha) = \infty$.

Theorem 1.1 was proved, first in [Dy04e].

CHAPTER 8

Solutions w_Γ are σ-moderate

In this chapter we consider the equation

$$\Delta u = u^\alpha, \quad 1 < \alpha \le 2$$

in a bounded domain E of class C^4 in $\mathbb{R}^d$ with $d \ge 4$. We prove a series of theorems leading to the equation $w_\Gamma = u_\Gamma$ for every Borel subset Γ of ∂D. (Recall that u_Γ and w_Γ are defined in Chapter **1** by (1.4), (1.5) and (1.6).)

1. Plan of the chapter

For every closed subset K of ∂E we put

$$\begin{aligned} E_\kappa(K) =& \{x \in E : d(x,K) \ge \kappa \operatorname{diam}(K)\}, \\ \varphi(x,K) =& \rho(x) d(x,K)^{-d}, \\ B_n(x,K) =& \{z : |x-z| < n d(x,K)\} \end{aligned} \tag{1.1}$$

where $\rho(x) = d(x, \partial E)$. We prove:

THEOREM 1.1. *For every $\kappa > 0$ there exists a constant C_κ such that, for every closed $K \subset \partial E$ and every $x \in E_\kappa(K)$,*

$$w_K(x) \le C_\kappa [\varphi(x,K)^\alpha \operatorname{Cap}_x(K)]^{1/(\alpha-1)}. \tag{1.2}$$

THEOREM 1.2. *There exist constants $C_\kappa > 0$ and n_κ such that, for every closed subset K of ∂E and for all $x \in E_\kappa(K)$, $\nu \in \mathcal{P}(K)$, subject to the condition $\mathcal{E}_x(\nu) < \infty$, we have*

$$\mathbb{N}_x\{\mathcal{R}_E \subset B_{n_\kappa}(x,K), Z_\nu\} \ge C_\kappa \varphi(x,K). \tag{1.3}$$

THEOREM 1.3. *There exist constants $C_\kappa > 0$ and $n(\kappa)$ with the property: for every closed $K \subset \partial E$ and for every $x \in E_\kappa(K)$,*

$$\mathbb{N}_x\{\mathcal{R}_E \subset B_{2n(\kappa)}(x,K), Z_\nu \ne 0\} \ge C_\kappa [\varphi(x,K)^\alpha \operatorname{Cap}_x(K)]^{1/(\alpha-1)} \tag{1.4}$$

for some $\nu \in \mathcal{P}(K)$ such that $\mathcal{E}_x(\nu) < \infty$.

THEOREM 1.4. *There exist constants C_κ and $n(\kappa)$ such that, for every closed $K \subset \partial E$ and every $x \in E_\kappa(K)$, there is a $\nu \in \mathcal{P}(K)$ with the properties: $\mathcal{E}_x(\nu) < \infty$ and*

$$w_K(x) \le C_\kappa \mathbb{N}_x\{\mathcal{R}_E \subset B_{2n(\kappa)}(x,K), Z_\nu \ne 0\}. \tag{1.5}$$

THEOREM 1.5. *There exists a constant C with the following property: for every closed $K \subset \partial E$ and every $x \in E$ there is a measure $\nu \in \mathcal{M}(K)$ such that $\mathcal{E}_x(\nu) < \infty$ and*

$$w_K(x) \le C\mathbb{N}_x\{Z_\nu \ne 0\}. \tag{1.6}$$

THEOREM 1.6. *For every closed $K \subset \partial E$, w_K is σ-moderate and $w_K = u_K$.*

THEOREM 1.7. *For every Borel subset Γ of ∂E, $w_\Gamma = u_\Gamma$.*

Theorem 1.1 follows immediately from Theorem **6**.2.2 and Kuznetsov's bound

$$w_K(x) \le C\varphi(x,K)\operatorname{Cap}(K)^{1/(\alpha-1)} \tag{1.7}$$

proved in [Ku04].

In Section 2 we establish some properties of conditional Brownian motion which we use in Section 3 to prove Theorem 1.2. By using Theorem 1.2 and the basic inequality (Theorem **7**.1.1), we prove Theorem 1.3. Theorem 1.4 follows at once from Theorems 1.1 and 1.3. In Section 5 we deduce Theorem 1.5 from Theorem 1.4. In Section 6 we get Theorem 1.6 from Theorem 1.5 and we deduce Theorem 1.7 from Theorem 1.6.

2. Three lemmas on the conditional Brownian motion

LEMMA 2.1. *If $d > 2$, then*

$$\hat{\Pi}_x^y \tau_E \le C|x-y|^2 \quad \text{for all } x \in E, y \in \partial E. \tag{2.1}$$

PROOF. We have

$$\hat{\Pi}_x^y\{t < \tau_E\} = \int_E \hat{p}_t(x,z)dz$$

where $\hat{p}_t(x,z)$ is the transition density of the conditional diffusion $(\xi_t, \hat{\Pi}_x^y)$. Therefore

$$\hat{\Pi}_x^y \tau_E = \hat{\Pi}_x^y \int_0^\infty 1_{t<\tau_E} dt = \int_0^\infty dt \int_E \hat{p}_t(x,z)dz = \int_E dz \int_0^\infty \hat{p}_t(x,z)dt.$$

Since $\hat{p}_t(x,z) = p_t(x,z)k_E(z,y)/k_E(x,y)$, we have

$$\hat{\Pi}_x^y \tau_E = k_E(x,y)^{-1} \int_E dz g_E(x,z) k_E(z,y). \tag{2.2}$$

We use estimates **2**.(1.6) for g_E and **2**.(1.10) for k_E. Since $\rho(z) \le |z-y|$ for $z \in E, y \in K$, it follows from (2.2) that

$$\hat{\Pi}_x^y \tau_E \le C|x-y|^d I \tag{2.3}$$

where

$$I = \int_{|z-y|\le R} |x-z|^{-a}|z-y|^{-b}\, dz$$

with $R = \operatorname{diam}(E), a = b = d-1$. Since $d-a-b = 2-d < 0$ for $d > 2$, $I \le C|x-y|^{2-d}$. [See, e.g., [La77], formula 1.1.3.] Therefore (2.1) follows from (2.3). □

The following lemma is proved in the Appendix A.

LEMMA 2.2. *For every $x \in E$,*

$$\Pi_x\{\sup_{t\le\tau_E} |\xi_t - x| \ge r\} \le C\rho(x)/r. \tag{2.4}$$

We need also the following lemma.

LEMMA 2.3. *Let $r = n\delta$ where $\delta = d(x,K)$ and let $\tau^r = \inf\{t : |\xi_t - x| \ge r\}$. There exist constants C_κ and s_κ such that*

(2.5)
$$\hat\Pi_x^y\{\tau^r < \tau_E\} \le C_\kappa(n-s_\kappa)^{-d} \quad \text{for all } x \in E_\kappa(K), y \in K \quad \text{and all } n > s_\kappa.$$

.

PROOF. It follows from (2.4) that

$$\Pi_x\{\tau^r < \tau_E\} \le C\rho(x)/r. \tag{2.6}$$

Put $\eta_r = \xi_{\tau^r}$. By **3**.1.3.B applied to $h(x) = k_E(x,y)$ and $\tau = \tau^r$,

$$\hat\Pi_x^y\{\tau^r < \tau_E\} = k_E(x,y)^{-1}\Pi_x\{\tau^r < \tau_E, k_E(\eta_r,y)\}. \tag{2.7}$$

By **2**.(1.10),

$$k_E(\eta_r,y) \le C\rho(\eta_r)|\eta_r - y|^{-d}. \tag{2.8}$$

If $y \in K, x \in E_\kappa(K)$, then

$$|x-y| \le d(x,K) + \operatorname{diam}(K) \le s_\kappa\delta \tag{2.9}$$

where $s_\kappa = 1 + 1/\kappa$. Therefore

$$|\eta_r - y| \ge |\eta_r - x| - |x-y| = r - |x-y| \ge r - s_\kappa\delta. \tag{2.10}$$

We also have

$$\rho(\eta_r) \le d(\eta_r,K) \le |\eta_r - x| + d(x,K) = r + \delta. \tag{2.11}$$

If $n > s_\kappa$, then, by (2.8), (2.10) and (2.11),

$$k_E(\eta_r,y) \le C(r+\delta)(r-s_\kappa\delta)^{-d}. \tag{2.12}$$

By **2**.(1.10) and (2.9),

$$k_E(x,y) \ge C'\rho(x)(s_\kappa\delta)^{-d}. \tag{2.13}$$

Formula (2.5) follows from (2.7), (2.12), (2.13) and (2.6). □

3. Proof of Theorem 1.2

1°. Put $B_m = B_m(x,K), U_m = B_m \cap E$. By Lemma **6**.3.2, there exists a smooth domain D such that $U_{2m} \subset D \subset U_{3m}$. By Theorem **7**.4.2,

$$\mathbb{N}_x\{\mathcal{R}_E \subset D^*, Z_\nu\} = I_x^\nu \tag{3.1}$$

where

$$I_x^\nu = \Pi_x^\nu\{A(D), e^{-\Phi(w_\Lambda)}\} \tag{3.2}$$

with

$$A(D) = \{\tau_D = \tau_E\}, \quad w_\Lambda(x) = \mathbb{N}_x\{\mathcal{R}_D \cap \Lambda \neq \emptyset\}. \tag{3.3}$$

Note that

$$I_x^\nu = \int_K k_E(x,y) I_x^y \nu(dy) \tag{3.4}$$

where

$$I_x^y = \hat{\Pi}_x^y\{A(D), e^{-\Phi(w_\Lambda)}\}.$$

Clearly, $A(U_m) \subset A(D)$ and therefore

$$I_x^y \geq \hat{\Pi}_x^y\{A(U_m), e^{-\Phi(w_\Lambda)}\}.$$

Since $e^{-t} \geq e^{-1} 1_{t \leq 1}$ for $t \geq 0$, we get

$$I_x^y \geq e^{-1}\hat{\Pi}_x^y\{A(U_m), \Phi(w_\Lambda) \leq 1\} = e^{-1}(1 - J_x^y - L_x^y) \tag{3.5}$$

where

$$J_x^y = \hat{\Pi}_x^y\{A(U_m), \Phi(w_\Lambda) > 1\}, \quad L_x^y = \hat{\Pi}_x^y[A(U_m)^c]. \tag{3.6}$$

2°. The next step is to obtain upper bounds for J_x^y and L_x^y.

We claim that

$$w_\Lambda(z) \leq C_d d(z, \partial B_{2m})^{-2/(\alpha-1)} \quad \text{for } z \in U_{2m}. \tag{3.7}$$

Indeed, the function

$$u(z) = \mathbb{N}_z\{\mathcal{R} \cap B_{2m}^c \neq \emptyset\} = -\log P_z\{\mathcal{R} \subset B_{2m}\}$$

belongs to $\mathcal{U}(B_{2m})$ and, by **2**.2.2.G,

$$u(z) \leq C d(z, \partial B_{2m})^{-2/(\alpha-1)} \quad \text{for } z \in B_{2m}.$$

This implies (3.7) because $\mathcal{R}_D \subset \mathcal{R}$ and $\Lambda \subset B_{2m}^c$ and, consequently, $w_\Lambda \leq u$.

Note that

$$J_x^y \leq \hat{\Pi}_x^y\{A(U_m), \Phi(w_\Lambda)\}. \tag{3.8}$$

If $z \in U_m$, then $d(z, B_{2m}^c) \geq m d(x,K)$ and, by (3.7),

$$w_\Lambda(z) \leq C[m d(x,K)]^{-2/(\alpha-1)}$$

. This implies

$$\Phi(w_\Lambda) \leq C[m d(x,K)]^{-2} \tau_E. \tag{3.9}$$

By Lemma 2.1 and (2.9),

(3.10) $\hat{\Pi}^y_x \tau_E \le C|x-y|^2 \le C(1+1/\kappa)^2 d(x,K)^2 \quad \text{for } y \in K, x \in E_\kappa(K).$

It follows from (3.8), (3.9) and (3.10) that

(3.11) $$J^y_x \le C_\kappa m^{-2} \quad \text{for } y \in K, x \in E_\kappa(K)$$

with $C_\kappa = C(1+1/\kappa)^2$.

3°. We have $A(U_m)^c = \{\tau_{U_m} < \tau_E\} = \{\tau^r < \tau_E\}$ where $r = m\delta$ and $\tau^r = \inf\{t : |\xi_t - x| \ge r\}$. By (3.6) and Lemma 2.3,

(3.12)
$$L^y_x = \hat{\Pi}^y_x\{\tau_r < \tau_E\} \le C_\kappa (m - s_\kappa)^{-d} \quad \text{for all } y \in K, x \in E_\kappa(K), m > s_\kappa.$$

4°. By (3.5), (3.11) and (3.12),

(3.13) $$I^y_x \ge C_{\kappa,m} \quad \text{for all } y \in K, x \in E_\kappa(K), m > s_\kappa$$

where

$$C_{\kappa,m} = e^{-1}[1 - C_\kappa m^{-2} - C_\kappa (m - s_\kappa)^{-d}].$$

5°. Note that $B_{4m} \supset \bar{B}_{3m} \supset \bar{D} \supset D^*$ and, by (3.1),

(3.14) $$\mathbb{N}_x\{\mathcal{R}_E \subset B_{4m}, Z_\nu\} \ge I^\nu_x.$$

By **2**.(1.10) and (2.9),

(3.15) $$k_E(x,y) \ge C^{-1} s_\kappa^{-d} \varphi(x,K) \quad \text{for all } x \in E_\kappa(K), y \in K.$$

By (3.14), (3.4), (3.13)and (3.15),

$$\mathbb{N}_x\{\mathcal{R}_E \subset B_{4m}, Z_\nu\} \ge C'_{\kappa,m} \varphi(x,K) \quad \text{for all } x \in E_\kappa(K), m > s_\kappa$$

where $C'_{\kappa,m} = C^{-1} s_\kappa^{-d} C_{\kappa,m}$. Note that $C'_{\kappa,m} \to C'_\kappa/e$ as $m \to \infty$ with $C'_\kappa = C^{-1} s_\kappa^{-d}$. Therefore there exists m_κ such that

$$\mathbb{N}_x\{\mathcal{R}_E \subset B_{4m_\kappa}, Z_\nu\} \ge \frac{1}{3} C'_\kappa \varphi(x,K) \quad \text{for all } x \in E_\kappa(K).$$

This implies (1.3) with $n_\kappa = 4m_\kappa$. □

4. Proof of Theorem 1.3

The relation (1.4) is trivial in the case $\mathrm{Cap}_x(K) = 0$. Suppose $\mathrm{Cap}_x(K) > 0$. It follows from **6**.(1.3) that, for some $\nu \in \mathcal{P}(K)$,

(4.1) $$\mathcal{E}_x(\nu)^{-1} \ge \mathrm{Cap}_x(K)/2.$$

For this ν, $\mathcal{E}_x(\nu) \le 2\,\mathrm{Cap}_x(K)^{-1} < \infty$.

We use notation B_m, U_m introduced in the proof of Theorem 1.2. Suppose that (1.3) holds for n_κ and C_κ and consider a smooth open set D such that $U_{2n_\kappa} \subset D \subset U_{4n_\kappa}$. By the basic inequality **7**.(1.1),

(4.2)
$$\mathbb{N}_x\{\mathcal{R}_E \subset D^*, Z_\nu \ne 0\} \ge C(\alpha)\mathbb{N}_x\{\mathcal{R}_E \subset D^*, Z_\nu\}^{\alpha/(\alpha-1)} \mathcal{E}_x(\nu)^{-1/(\alpha-1)}$$

if ν is concentrated on $\partial E \cap \partial D$ and if $\mathcal{E}_x(\nu) < \infty$. Therefore, by (4.1), there exists ν supported by K such that $\mathcal{E}_x(\nu) < \infty$ and

(4.3)
$$\mathbb{N}_x\{\mathcal{R}_E \subset D^*, Z_\nu \neq 0\} \geq C(\alpha)\mathbb{N}_x\{\mathcal{R}_E \subset D^*, Z_\nu\}^{\alpha/(\alpha-1)} \operatorname{Cap}_x(K)^{1/(\alpha-1)}.$$

We have $D^* \subset B_{4n_\kappa}$ (cf. part 5° in the proof of Theorem 1.2). Note that $B_{n_\kappa} \cap \bar{E} \subset D^*$ and therefore, if $\mathcal{R}_E \subset B_{n_\kappa}$, then $\mathcal{R}_E \subset D^*$. Thus (4.3) implies

(4.4)
$$\mathbb{N}_x\{\mathcal{R}_E \subset B_{4n_\kappa}, Z_\nu \neq 0\} \geq C(\alpha)\mathbb{N}_x\{\mathcal{R}_E \subset B_{n_\kappa}, Z_\nu\}^{\alpha/(\alpha-1)} \operatorname{Cap}_x(K)^{1/(\alpha-1)}.$$

The bound (1.4) with $n(\kappa) = 4n_\kappa$ follows from (4.4) and (1.3). □

5. Proof of Theorem 1.5

Put

$$V_m = B_{2^m}(x, K),$$
$$K_1 = K \cap \bar{V}_1 = \{z \in K : |x - z| \leq 2d(x, K)\},$$
$$K_m = K \cap (\bar{V}_m \setminus V_{m-1}) = \{z \in K : 2^{m-1}d(x, K) \leq |x - z| \leq 2^m d(x, K)\} \quad \text{for } m > 1.$$

Note that

$$\operatorname{diam}(K_m) \leq \operatorname{diam}(V_m) = 2^{m+1}d(x, K)$$

and

$$d(x, K_m) \geq d(x, \partial V_{m-1}) = 2^{m-1}d(x, K) \geq \frac{1}{4}\operatorname{diam}(K_m)$$

and therefore $x \in E_\kappa(K_m)$ with $\kappa = 1/4$. By Theorem 1.4 applied to K_m, there is $\nu_m \in \mathcal{P}(K_m)$ with the properties: $\mathcal{E}_x(\nu_m) < \infty$ and

$$w_{K_m}(x) \leq C_\kappa \mathbb{N}_x\{\mathcal{R}_E \subset B_{2n(\kappa)}(x, K_m), Z_{\nu_m} \neq 0\}. \tag{5.1}$$

We have $d(x, K_m) \leq d(x, \partial V_m) = 2^m d(x, K)$ and therefore, if $2^p \geq 2n(\kappa)$, then for every positive integer m,

$$B_{2n(\kappa)}(x, K_m) \subset B_{2^{p+m}}(x, K) = V_{p+m}.$$

By (5.1),

$$w_{K_m}(x) \leq C_\kappa \mathbb{N}_x(Q_m) \tag{5.2}$$

where

$$Q_m = \{\mathcal{R}_E \subset V_{p+m}, Z_{\nu_m} \neq 0\}. \tag{5.3}$$

We claim that

$$\mathbb{N}_x(Q_m \cap Q_{m'}) = 0 \quad \text{for } m' \geq m + p + 1. \tag{5.4}$$

First, we note that $K_{m'} \subset \partial V_{m'}$ and therefore $K_{m'} \cap V_{m+p} = \emptyset$ because $K_{m'} \cap V_{m'} = \emptyset$. Next, we observe that

$$Q_m \cap Q_{m'} \subset \{\mathcal{R}_E \subset V_{p+m}, Z_{\nu_{m'}} \neq 0\} \subset \{\mathcal{R}_E \cap K_{m'} = \emptyset, Z_{\nu_{m'}} \neq 0\}.$$

Since $\nu_{m'}$ is concentrated on $K_{m'}$, (5.4) follows from **4.**(3.30).

If $K_m = \emptyset$, then $\nu_m = 0$ satisfies (5.2). There exist only a finite number of m for which K_m is not empty. Therefore

$$\nu = \sum_1^\infty \nu_m$$

is a finite measure concentrated on K and $\mathcal{E}_x(\nu) \le \sum \mathcal{E}_x(\nu_m) < \infty$. [1]

By **4**.(3.19),

$$w_K(x) = \mathbb{N}_x\{\mathcal{R}_E \cap K \ne \emptyset\}$$

and therefore

$$w_K(x) \le \sum_1^\infty \mathbb{N}_x\{\mathcal{R}_E \cap K_m \ne \emptyset\} = \sum_1^\infty w_{K_m}(x).$$

By (5.2), this implies

$$w_K(x) \le C_\kappa \sum_{m=1}^\infty \mathbb{N}_x(Q_m). \tag{5.5}$$

Every integer $m \ge 1$ has a unique representation $m = n(p+1) + j$ where $j = 1, \dots, p+1$ and therefore

$$w_K(x) \le C_\kappa \sum_{j=1}^{p+1} \sum_{n=0}^\infty \mathbb{N}_x(Q_{n(p+1)+j}). \tag{5.6}$$

It follows from (5.4) that $\mathbb{N}_x\{Q_{n(p+1)+j} \cap Q_{n'(p+1)+j}\} = 0$ for $n' > n$. Therefore, for every j,

$$\begin{aligned} \sum_{n=0}^\infty \mathbb{N}_x\{Q_{n(p+1)+j}\} &= \mathbb{N}_x\left\{\bigcup_{n=0}^\infty Q_{n(p+1)+j}\right\} \\ &\le \mathbb{N}_x\left\{\sum_{n=0}^\infty Z_{\nu_{n(p+1)+j}} \ne 0\}\right\} \le \mathbb{N}_x\{Z_\nu \ne 0\} \end{aligned} \tag{5.7}$$

because

$$\sum_{n=0}^\infty Z_{\nu_{n(p+1)+j}} \le \sum_{m=1}^\infty Z_{\nu_m} = Z_\nu.$$

The bound (1.6) (with $C = (p+1)C_\kappa$) follows from (5.6) and (5.7). □

6. Proof of Theorems 1.6 and 1.7

6.1. Proof of Theorem 1.6. By Theorem 1.5, for every $x \in E$, there exists $\nu = \nu_x \in \mathcal{M}(K)$ such that $\mathcal{E}_x(\nu_x) < \infty$ and

$$w_K(x) \le C\mathbb{N}_x\{Z_{\nu_x} \ne 0\}. \tag{6.1}$$

[1]Measures ν_m and ν depend on K and x.

Consider a countable set Λ everywhere dense in E and put

$$\mu = \sum_{x\in\Lambda} \nu_x.$$

By **2**.2.3.E, the condition $\mathcal{E}_x(\nu_x) < \infty$ implies that $\nu_x \in \mathcal{N}_1^E$. By the definition of $\mathcal{N}_0^E$, this class contains μ and $\eta = \infty\cdot\mu$. Since η does not charge $\partial E\setminus K$, $u_\eta = 0$ on $\partial E\setminus K$ by **2**.2.4.C and

$$u_\eta \le w_K \tag{6.2}$$

by **1**.(1.5). By (6.1) and **4**.(3.32),

$$w_K(x) \le C\mathbb{N}_x\{Z_{\nu_x}\ne 0\} \le C\mathbb{N}_x\{Z_\eta \ne 0\} = Cu_\eta(x) \quad \text{for } x\in\Lambda. \tag{6.3}$$

Since w_K and u_η are continuous, (6.3) holds for all $x\in E$ and therefore $Z_{w_K}\le CZ_{u_\eta}$. Since $C\eta=\eta$ for all $C>0$, we have $CZ_\eta = Z_{C\eta} = Z_\eta$. Hence $Z_{w_K}\le Z_\eta$. By **3**.(3.4), this implies $w_K\le u_\eta$ and, by (6.2), $w_K = u_\eta$. We conclude that w_K is σ-moderate.

By **1**.(1.4)-(1.5), $u_\eta \le u_K \le w_K$. Hence $u_K = w_K$. □

6.2. Proof of Theorem 1.7. If K is a compact subset of a Borel set Γ, then, by Theorem 1.6,

$$w_K = u_K \le u_\Gamma.$$

By **1**.(1.6), this implies $w_\Gamma \le u_\Gamma$.

On the other hand, if ν is concentrated on Γ, then, by **2**.2.5.B, $u_\nu \le w_\Gamma$ and, by **1**.(1.4), $u_\Gamma\le w_\Gamma$.

7. Notes

The general plan of this chapter is close to the plan of Chapter 3 of Mselati's thesis. To implement this plan in the case of equation $\Delta u = u^\alpha$ with $\alpha\ne 2$ we need the enhancements of the superdiffusion theory in Chapters **4**, **5**, **6** and **7**. Some of Mselati's arguments are used with very little modification. In particular, our proof of Theorem 1.2 is close to his proof of Lemma 3.2.2 and the proof of Theorem 1.5 is based on the construction presented on pages 94-95 in [Ms02a] and pages 81-82 in [Ms04].

Kuznetsov's upper bound for w_K is a generalization of the bound obtained by Mselati for $\alpha = 2$ in Chapter 3 of [Ms02a].

We left aside the case $d = 3$. [2] It can be covered on the price of a complication of the formulae. Mselati has done this for $\alpha = 2$ and his arguments can be adjusted to $\alpha < 2$.

In [MV04] Marcus and Véron proved that $w_K = u_K$ in the case of a domain E of class C^2 and the equation $\Delta u = u^\alpha$ for all $\alpha > 1$ (not only for $1<\alpha\le 2$). [3] To this end they establish upper and lower capacitary bounds for w_K but they use not the Poisson capacity but the Bessel capacity $C_{2/\alpha,\alpha'}$

[2]It is well-known that for $d < 3$ all solutions are σ-moderate and therefore we do not need to consider these dimensions.

[3]The result was announced in [MV03].

on ∂E [which also belongs to the class of capacities defined in Section 1 of Chapter 6.] The relations between this capacity and the Poisson capacity proved in the Appendix B imply that the capacitary bounds in [MV04] are equivalent to the bounds used in the present book.

The paper [MV04] contains also results on asymptotic behavior of w_K at points of K.

CHAPTER 9

All solutions are σ-moderate

To complete the program described in the Introduction (see Section **1**.2) it remains to prove that, if $\mathrm{Tr}(u) = (\Gamma, \nu)$, then $u \le w_\Gamma \oplus u_\nu$. To get this result, it is sufficient to prove:

A. Our statement is true for a domain E if, for every $y \in \partial E$, there exists a domain $D \subset E$ for which it is true such that $\partial D \cap \partial E$ contains a neighborhood of y in ∂E.

B. The statement is true for star domains.

[A domain E is called a *star domain* relative to a point c if, for every $x \in E$, the line segment $[c, x]$ connecting c and x is contained in E.]

1. Plan

Our goal is to prove:

THEOREM 1.1. *If u is a positive solution of the equation*

$$\Delta u = u^\alpha \quad \text{in } E \tag{1.1}$$

where $1 < \alpha \le 2$ and E is a bounded domain of class C^4 and if $\mathrm{Tr}(u) = (\Gamma, \nu)$, then

$$u \le w_\Gamma \oplus u_\nu. \tag{1.2}$$

Recall that, by **1**.1.5.B,

$$u_\Gamma \oplus u_\nu \le u \tag{1.3}$$

and, by Theorem **8**.1.7,

$$w_\Gamma = u_\Gamma. \tag{1.4}$$

Thus it follows from Theorem 1.1 that

$$u = u_\Gamma \oplus u_\nu = w_\Gamma \oplus u_\nu \tag{1.5}$$

and u is σ-moderate because so are u_Γ and u_ν.

Denote by $\mathfrak{E}$ the class of domains for which Theorem 1.1 is true and by $\mathfrak{E}_1$ the class of domains with the property:

1.A. If $\mathrm{Tr}(u) = (\Lambda, \nu), \Lambda \subset \Gamma \subset \partial E$ and $\nu(\partial E \setminus \Gamma) = 0$, then $u \le w_\Gamma$.

PROPOSITION 1.1.

$$\mathfrak{E}_1 \subset \mathfrak{E}.$$

PROOF. Suppose that $E \in \mathfrak{E}_1$ and $\mathrm{Tr}(u) = (\Gamma, \nu)$. By the definition of the trace, $u_\nu \le u$ (see **1**.(1.7)). We will prove (1.2) by applying 1.A to $v = u \ominus u_\nu$.

Let $\mathrm{Tr}(v) = (\Lambda, \mu)$. Clearly, $\Lambda \subset \Gamma$. If we show that $\mu(\partial E \setminus \Gamma) = 0$, then 1.A will imply that $v \le w_\Gamma$ and therefore $v \oplus u_\nu \le w_\Gamma \oplus u_\nu$. By Lemma **3**.3.1, $v \oplus u_\nu = u$.

It remains to prove that $\mu(\partial E \setminus \Gamma) = 0$. By the definition of the trace,

$$\mu(\partial E \setminus \Gamma) = \sup\{\lambda(\partial E \setminus \Gamma) : \lambda \in \mathcal{N}_1^E, \lambda(\Gamma) = 0, u_\lambda \le v\}. \tag{1.6}$$

Since $\nu(\Gamma) = 0$ and $\nu \in \mathcal{N}_1^E$, the conditions $\lambda \in \mathcal{N}_1^E, \lambda(\Gamma) = 0$ imply $(\lambda + \nu)(\Gamma) = 0$, $\lambda + \nu \in \mathcal{N}_1$. By Lemma **3**.3.2, $u_{\lambda+\nu} = u_\lambda \oplus u_\nu$ and, $u_{\lambda+\nu} \le v \oplus u_\nu = u$ because $u_\lambda \le v$. By **1**.(1.7), $\lambda + \nu \le \nu$. Hence $\lambda = 0$ and $\mu(\partial E \setminus \Gamma) = 0$ by (1.6). □

1.1. In Section 2 we prove the following Localization theorem:

THEOREM 1.2. *E belongs to $\mathfrak{E}_1$ if , for every $y \in \partial E$, there exists a domain $D \in \mathfrak{E}_1$ such that $D \subset E$ and $\partial D \cap \partial E$ contains a neighborhood of y in ∂E.*

Theorem 1.1 follows from Proposition 1.1, Theorem 1.2 and the following theorem which will be proved in Section 3:

THEOREM 1.3. *The class $\mathfrak{E}_1$ contains all star domains.*

2. Proof of Localization theorem

2.1. Preparations. Suppose that D is a smooth subdomain of a bounded smooth domain E. Put $L = \{x \in \partial D : d(x, E \setminus D) > 0\}$.

We need the following lemmas.

LEMMA 2.1. *If a measure $\nu \in \mathcal{N}_1^D$ is concentrated on L, then $\nu \in \mathcal{N}_1^E$.*

PROOF. For every $x \in D$, $P_x\{\mathcal{R}_E \supset \mathcal{R}_D\} = 1$ and therefore $K \subset L$ is $\mathcal{R}_D$-polar if it is $\mathcal{R}_E$-polar. If $\eta \in \mathcal{N}_1^D$, then $\eta(K) = 0$ for all $\mathcal{R}_D$-polar K. Hence $\eta(K) = 0$ for all $\mathcal{R}_E$-polar $K \subset L$. Since η is concentrated on L, it vanishes on all $\mathcal{R}_E$-polar K and it belongs to $\mathcal{N}_1^E$ by Theorem **3**.3.5. □

It follows from Lemma 2.1 that a moderate solution u_η in E and a moderate solution $\tilde{u}_\eta$ in D correspond to every $\eta \in N_1^D$ concentrated on L.

LEMMA 2.2. *Suppose that a measure $\eta \in \mathcal{N}_1^D$ is concentrated on a closed subset K of L. Let u_η be the maximal element of $\mathcal{U}(E)$ dominated by*

$$h_\nu(x) = \int_K k_E(x, y)\eta(dy) \tag{2.1}$$

and let $\tilde{u}_\eta$ be the maximal element of $\mathcal{U}(D)$ dominated by

$$\tilde{h}_\eta(x) = \int_K k_D(x, y)\eta(dy). \tag{2.2}$$

Then, for every $y \in L$,

$$\lim_{x\to y}[u_\eta(x) - \tilde{u}_\eta(x)] = 0. \tag{2.3}$$

PROOF. It follows from **3**.1.1.C that

$$h_\eta(x) = \tilde{h}_\eta(x) + \Pi_x 1_{\tau_D<\tau_E} h_\eta(\xi_{\tau_D}). \tag{2.4}$$

This implies $h_\eta \geq \tilde{h}_\eta$ and

$$h_\eta(x) - \tilde{h}_\eta(x) \to 0 \quad \text{as } x \to y. \tag{2.5}$$

The equation (2.3) will be proved if we show that

$$0 \leq u_\eta - \tilde{u}_\eta \leq h_\eta - \tilde{h}_\eta \quad \text{in } D. \tag{2.6}$$

Note that

$$u_\eta + G_E u_\eta^\alpha = h_\eta \quad \text{in } E, \tag{2.7}$$

$$\tilde{u}_\eta + G_D \tilde{u}_\eta^\alpha = \tilde{h}_\eta \quad \text{in } D \tag{2.8}$$

and

$$u_\eta + G_D u_\eta^\alpha = h' \quad \text{in } D \tag{2.9}$$

where h' is the minimal harmonic majorant of u_η in D. Hence

$$u_\eta - \tilde{u}_\eta = h_\eta - \tilde{h}_\eta - G_E u_\eta^\alpha + G_D \tilde{u}_\eta^\alpha \quad \text{in } D. \tag{2.10}$$

By (2.7), $G_E u_\eta^\alpha \leq h_\eta$ and therefore, by **3**.1.1.A and the strong Markov property of ξ,

$$\begin{aligned}(G_E - G_D)u_\eta^\alpha(x) &= \Pi_x \int_{\tau_D}^{\tau_E} u_\eta(\xi_s)^\alpha \, ds \\ &= \Pi_x 1_{\tau_D<\tau_E} G_E u_\eta^\alpha(\xi_{\tau_D}) \leq \Pi_x 1_{\tau_D<\tau_E} h_\eta(\xi_{\tau_D}) \quad \text{in } D.\end{aligned} \tag{2.11}$$

It follows from (2.4) and (2.11) that

$$h_\eta(x) - \tilde{h}_\eta(x) \geq (G_E - G_D)u_\eta^\alpha(x) \quad \text{in } D. \tag{2.12}$$

On the other hand, by (2.7) and (2.9),

$$(G_E - G_D)u_\eta^\alpha = h_\eta - h' \quad \text{in } D. \tag{2.13}$$

By (2.12) and (2.13), $\tilde{h}_\eta \leq h'$ in D. This implies $\tilde{u}_\eta \leq u_\eta$ in D and $G_D \tilde{u}_\eta^\alpha \leq G_D u_\eta^\alpha \leq G_E u_\eta^\alpha$. Formula (2.6) follows from (2.10). □

LEMMA 2.3. *Suppose that u' is the restriction of $u \in \mathcal{U}(E)$ to D and let*

$$\mathrm{Tr}(u) = (\Lambda, \nu), \quad \mathrm{Tr}(u') = (\Lambda', \nu') \tag{2.14}$$

We have

$$\Lambda' = \Lambda \cap \bar{L}. \tag{2.15}$$

If $\Gamma \supset \Lambda$ and $\nu(\partial E \setminus \Gamma) = 0$, then $\nu'(L \cap \Gamma^c) = 0$.

PROOF. 1°. If $y \in \partial D \cap E$, then, Π_x^y-a.s., $u'(\xi_t)$ is bounded on $[0, \tau_D)$and therefore $\Phi(u') < \infty$. Hence, $\Lambda' \subset \bar{L}$.

By Corollary **3**.1.1 to Lemma **3**.1.2,

(2.16)
$$\tilde{\Pi}_x^y\{\Phi(u') < \infty\} = \Pi_x^y\{\Phi(u') < \infty, \tau_D = \tau_E\} = \Pi_x^y\{\Phi(u) < \infty, \tau_D = \tau_E\}$$

for all $x \in D, y \in \bar{L}$. Therefore $\Lambda \cap \bar{L} \subset \Lambda'$. If $y \in \Lambda'$, then $\tilde{\Pi}_x^y\{\Phi(u') < \infty\} = 0$ and, since $y \in \bar{L}$, $\Pi_x^y\{\tau_D \neq \tau_E\} = 0$ for all $x \in D$. By (2.16), $\Pi_x^y\{\Phi(u) < \infty\} = 0$. Therefore $\Lambda' \subset \Lambda \cap \bar{L}$ which implies (2.15).

2°. Denote by $\mathcal{K}$ the class of compact subsets of L such that the restriction of ν' to K belongs to $\mathcal{N}_1^D$. To prove the second statement of the lemma, it is sufficient to prove that the condition

$$K \in \mathcal{K},\ \eta \leq \nu' \text{ and } \eta \text{ is concentrated on } K \tag{2.17}$$

implies that $\eta(L \cap \Gamma^c) = 0$. Indeed, by **1**.1.5.A, ν' is a σ-finite measure of class N_0^D. There exist Borel sets $B_m \uparrow \partial D$ such that $\nu'(B_m) < \infty$. Put $L_m = B_m \cap L$. We have $\nu'(L_m \setminus K_{mn}) < 1/n$ for some compact subsets K_{mn} of L_m. Denote by η_{mn} the restriction of ν' to K_{mn}. By **2**.2.4.B, $\eta_{mn} \in \mathcal{N}_0^D$ and, since $\eta_{mn}(\partial D) < \infty$, $\eta_{mn} \in \mathcal{N}_1^D$ by **2**.2.4.A. Hence $K_{mn} \in \mathcal{K}$. The pair (K_{mn}, η_{mn}) satisfies the condition (2.17). It remains to note that, if $\eta_{m,n}(L \cap \Gamma^c) = 0$ for all m, n, then $\nu'(L \cap \Gamma^c) = 0$.

3°. First, we prove that (2.17) implies

$$\eta \in \mathcal{N}_1^E, \quad \eta(\Lambda) = 0, \quad u_\eta \leq u. \tag{2.18}$$

Suppose that (2.17) holds. The definition of $\mathcal{K}$ implies that $\eta \in \mathcal{N}_1^D$. By Lemma 2.1, $\eta \in \mathcal{N}_1^E$. By (2.15), $\Lambda \subset \Lambda' \cup (\partial E \setminus \bar{L})$. Hence $\eta(\Lambda) = 0$ because $\eta(\Lambda') \leq \nu'(\Lambda') = 0$ and η is concentrated on $K \subset \bar{L}$. It remains to check that $u_\nu \leq u$. We have $\tilde{u}_\eta \leq \tilde{u}_{\nu'} \leq \tilde{u}_{\Lambda'} \oplus \tilde{u}_{\nu'}$ and therefore, by **1**.1.5.B, $\tilde{u}_\eta \leq u'$. Since $u_\eta(x) \leq h_\eta(x)$, we have

$$\lim_{x \to y} u_\eta(x) = 0 \leq u(x) \quad \text{for } y \in \partial E \setminus K.$$

By Lemma 2.2,

$$\limsup_{x \to y} [u_\eta(x) - u(x)] = \limsup_{x \to y} [u'_\eta(x) - u'(x)] \leq 0 \quad \text{for } y \in L$$

By the Comparison principle **2**.2.2.B, this implies $u_\eta \leq u$ in E.

4°. By **1**.(1.7), it follows from (2.18) that $\eta \leq \nu$ and therefore $\eta(L \cap \Gamma^c) \leq \nu(L \cap \Gamma^c) \leq \nu(\partial E \setminus \Gamma) = 0$. □

2.2. Proof of Theorem 1.2. We need to prove that, if $\mathrm{Tr}(u) = (\Lambda, \nu)$ and if $\nu(\Gamma^c) = 0$ where $\Lambda \subset \Gamma \subset \partial E$, then $u \leq w_\Gamma$.

The main step is to show that

$$\limsup_{x \to y} [u(x) - 2w_\Gamma(x)] \leq 0 \quad \text{for all } y \in \partial E. \tag{2.19}$$

Fix y and consider a domain $D \in \mathfrak{E}$ such that $D \subset E$ and $\partial D \cap \partial E$ contains a neighborhood of y in ∂E. We use the notation introduced in

Lemma 2.3. Clearly, $y \in L$. By the definition of $\mathfrak{E}$, **2**.3.A and **2**.2.5.B,

$$u' \le \tilde{w}_{\Lambda'} \oplus \tilde{u}_{\nu'} = \pi(\tilde{w}_{\Lambda'} + \tilde{u}_{\nu'}) \le \tilde{w}_{\Lambda'} + \tilde{u}_{\nu'} \le 2\tilde{w}_{\Lambda'}. \tag{2.20}$$

Note that $\Lambda' = \Lambda \cap \bar{L} \subset \Gamma \cap \bar{L} \subset (\Gamma \cap L) \cup A$ where A is the closure of $\partial D \cap E$. By **3**.3.5.C, this implies

$$\tilde{w}_{\Lambda'} \le \tilde{w}_{\Gamma \cap L} + \tilde{w}_A$$

and, by (2.20),

$$u' \le 2\tilde{w}_{\Gamma \cap L} + 2\tilde{w}_A. \tag{2.21}$$

Since $\mathcal{R}_D \subset \mathcal{R}_E$, **4**.(3.19) implies that, for every Borel subset B of $\bar{L}$,

$$\tilde{w}_B = \mathbb{N}_x\{\mathcal{R}_D \cap B \ne \emptyset\} \le \mathbb{N}_x\{\mathcal{R}_E \cap B \ne \emptyset\} = w_B \quad \text{on } D. \tag{2.22}$$

Thus $\tilde{w}_{\Gamma \cap L} \le w_{\Gamma \cap L} \le w_\Gamma$ and (2.21) implies $u' \le 2w_\Gamma + 2\tilde{w}_A$. Hence,

$$\limsup_{x \to y, x \in E} [u(x) - 2w_\Gamma(x)] = \limsup_{x \to y, x \in D} [u'(x) - 2w_\Gamma(x)] \le \limsup_{x \to y, x \in D} \tilde{w}_A(x).$$

By **2**.2.5.A, this implies (2.19). It follows from the Comparison principle, that $u \le 2w_\Gamma$ in E. Therefore $Z_u \le 2Z_\Gamma$ where $Z_\Gamma = \mathrm{SBV}(w_\Gamma)$. By **3**.3.5.A, $2Z_\Gamma = Z_\Gamma$ and, by **3**.(3.4), $u = \mathrm{LPT}(Z_u) \le \mathrm{LPT}(Z_\Gamma) = w_\Gamma$. □

3. Star domains

3.1. In this section we prove Theorem 1.3. Without any loss of generality we can assume that E is a star domain relative to $c = 0$.

We use the self-similarity of the equation

$$\Delta u = u^\alpha \quad \text{in } E. \tag{3.1}$$

Let $0 < r \le 1$. Put $E_r = rE$, $\beta = 2/(\alpha - 1)$ and

$$f_r(x) = r^\beta f(rx) \quad \text{for } x \in E, f \in \mathcal{B}(E). \tag{3.2}$$

If $u \in \mathcal{U}(E)$, then u_r also belongs to $\mathcal{U}(E)$. Moreover, for $r < 1$, u_r is continuous on $\bar{E}$ and $u_r \to u$ uniformly on each $D \Subset E$ as $r \uparrow 1$. If f is continuous, then, for every constant $k > 0$,

$$V_E(kf_r)(x) = r^\beta V_{E_r}(kf)(rx) \quad \text{for all } x \in E. \tag{3.3}$$

This is trivial for $r = 1$. For $r < 1$ this follows from **2**.2.2.A because both parts of (3.3) are solutions of the equation (3.1) with the same boundary condition $u = kf_r$ on ∂E.

3.2. Preparations.

LEMMA 3.1. *Every sequence $u_n \in \mathcal{U}(E)$ contains a subsequence u_{n_i} which converges uniformly on each set $D \Subset E$ to an element of $\mathcal{U}(E)$.*

PROOF. We use a gradient estimate for a solution of the Poisson equation $\Delta u = f$ in D (see [GT98], Theorem 3.9)

$$\sup_D(\rho|\nabla u|) \le C(D)(\sup_D |u| + \sup_D(\rho^2|f|)). \tag{3.4}$$

Suppose $D \Subset E$. By **2**.2.2.E, there exists a constant b such that all $u \in \mathcal{U}(E)$ do not exceed b in D. By (3.4),

$$\sup_D(\rho|\nabla u|) \le C(D)(b + \operatorname{diam}(D)^2 b^\alpha) = C'(D).$$

If $\tilde{D} \Subset D$, then there exists a constant $a > 0$ such that $|x - y| \ge a$ for all $x \in \tilde{D}, y \in \partial D$. Therefore, for all $x \in \tilde{D}$, $\rho(x) = d(x, \partial D) \ge a$ and $|\nabla u|(x) \le C'(D)/a$. The statement of the lemma follows from Arzela's theorem (see, e.g., [Ru87], Theorem 11.28). □

LEMMA 3.2. *Put*

$$Y_r = \exp(-Z_{u_r}). \tag{3.5}$$

For every $\gamma \ge 1$,

$$P_0|Y_r - Y_1|^\gamma \to 0 \quad \text{as } r \uparrow 1. \tag{3.6}$$

PROOF. 1°. First we prove that

$$\lim_{r\uparrow 1} P_0(Y_r^k - Y_1^k)^2 = 0 \tag{3.7}$$

for every positive integer k. If (3.7) does not hold, then

$$\lim P_0(Y_{r_n}^k - Y_1^k)^2 > 0 \tag{3.8}$$

for some sequence $r_n \uparrow 1$.

Note that

$$P_0(Y_r^k - Y_1^k)^2 = F_r + F_1 - 2G_r \tag{3.9}$$

where $F_r = P_0 Y_r^{2k}, G_r = P_0(Y_r Y_1)^k$. By **3**.(2.6) and (3.3),

(3.10)
$$F_r = P_0 \exp[-2k\langle u_r, X_E\rangle] = \exp[-V_E(2ku_r)(0)] = \exp[-r^\beta V_{E_r}(2ku)(0)]$$
$$= \{\exp[-V_{E_r}(2ku)(0)]\}^{r^\beta} = \{P_0 \exp(-2k\langle u, X_{E_r}\rangle)\}^{r^\beta}.$$

Since $\langle u, X_{E_{r_n}}\rangle \to Z_u$ P_0-a.s., we have

$$F_{r_n} \to F_1. \tag{3.11}$$

By (3.10) and (3.11),

$$P_0 e^{-2k\langle u, X_{E_{r_n}}\rangle} \to F_1. \tag{3.12}$$

Put

$$v_r(x) = -\log P_x(Y_r Y_1)^k = -\log P_x \exp[-k(Z_{u_r} + Z_u)]. \tag{3.13}$$

By **3**.3.4.A, $k(Z_{u_r} + Z_u) \in \mathfrak{Z}$ and

$$v_r \le k(u_r + u) \quad \text{in } E. \tag{3.14}$$

By Theorem **3**.3.3, $v_r \in \mathcal{U}(E)$. By Lemma 3.1, we can choose a subsequence of the sequence v_{r_n} that converges uniformly on each $D \Subset E$ to an element v of $\mathcal{U}(E)$. By changing the notation we can assume that this subsequence coincides with the sequence v_{r_n}. By **3**.(3.4), $P_x e^{-Z_v} = e^{-v(x)}$ and therefore

$$G_{r_n} = e^{-v_{r_n}(0)} \to e^{-v(0)} = P_0 e^{-Z_v}. \tag{3.15}$$

By passing to the limit in (3.14), we get that $v \le 2ku$. Therefore $Z_v \le 2kZ_u$ and

$$P_0 e^{-Z_v} \ge P_0 e^{-2kZ_u} = \lim P_0 e^{-2k\langle u, X_{E_{r_n}}\rangle}. \tag{3.16}$$

It follows from (3.15), (3.16) and (3.12), that $\lim G_{r_n} \ge F_1$. Because of (3.9) and (3.11), this contradicts (3.8).

2°. If $\gamma < m$, then $(P_0|Z|^\gamma)^{1/\gamma} \le (P_0|Z|^m)^{1/m}$. Therefore it is sufficient to prove (3.6) for even integers $\gamma = m > 1$. Since $0 \le Y_1 \le 1$, the Schwarz inequality and (3.7) imply

$$P_0|Y_r^k Y_1^{m-k} - Y_1^m| \le (P_0 Y_1^{2(m-k)})^{1/2} [P_0(Y_r^k - Y_1^k)^2]^{1/2} \to 0 \quad \text{as } r \uparrow 1.$$

Therefore

$$P_0|Y_r - Y_1|^m = P_0(Y_r - Y_1)^m = \sum_{k=0}^m \binom{m}{k} (-1)^{m-k} P_0(Y_r)^k Y_1^{m-k}$$
$$\to \sum_{k=0}^m \binom{m}{k} (-1)^{m-k} P_0 Y_1^m = 0.$$

□

LEMMA 3.3. *For every $\nu \in \mathcal{N}_1^E$, for every $1 < \gamma < \alpha$ and for all $x \in E$,*

$$P_x Z_\nu^\gamma \le 1 + c_1 h_\nu(x)^2 + c_2 G_E(h_\nu^\alpha)(x) \tag{3.17}$$

where $c_1 = \frac{1}{2} e\gamma/(2-\gamma)$ and $c_2 = e\gamma/(\alpha - \gamma)$.

PROOF. For every probability measure P and for every positive Z

$$\begin{aligned} PZ^\gamma &= P\int_0^Z \gamma\lambda^{\gamma-1} d\lambda \\ &= \int_0^\infty P\{Z > \lambda\}\gamma\lambda^{\gamma-1} d\lambda \le 1 + \int_1^\infty P\{Z > \lambda\}\gamma\lambda^{\gamma-1} d\lambda. \end{aligned} \tag{3.18}$$

Function

$$\mathcal{E}(\lambda) = e^{-\lambda} - 1 + \lambda, \quad \lambda > 0$$

is positive, monotone increasing and $\mathcal{E}(1) = 1/e$. For each $\lambda > 0$, by Chebyshev's inequality,

$$P\{Z > \lambda\} = P\{Z/\lambda > 1\} = P\{\mathcal{E}(Z/\lambda) > 1/e\} \le eq(1/\lambda) \tag{3.19}$$

where $q(\lambda) = P\mathcal{E}(\lambda Z)$. By (3.18) and (3.19),

$$PZ^\gamma \le 1 + e\int_0^1 \gamma\lambda^{-\gamma-1} q(\lambda) d\lambda. \tag{3.20}$$

We apply (3.20) to $P = P_x$ and to $Z = Z_\nu$. By **3**.(3.13) and **3**.3.6.B,

$$q(\lambda) = P_x e^{-\lambda Z_\nu} - 1 + \lambda P_x Z_\nu \\ = e^{-u_{\lambda\nu}(x)} - 1 + \lambda h_\nu(x) = \mathcal{E}(u_{\lambda\nu}) + \lambda h_\nu - u_{\lambda\nu}. \tag{3.21}$$

Since $\mathcal{E}(\lambda) \le \frac{1}{2}\lambda^2$, we have

$$\mathcal{E}(u_{\lambda\nu})(x) \le \frac{1}{2} u_{\lambda\nu}(x)^2 \le \frac{1}{2}\lambda^2 h_\nu(x)^2. \tag{3.22}$$

By **3**.3.6.B,

$$\lambda h_\nu - u_{\lambda\nu} = G_E(u_{\lambda\nu}^\alpha) \le \lambda^\alpha G_E(h_\nu^\alpha). \tag{3.23}$$

Formula (3.17) follows from (3.20), (3.21) (3.22) and (3.23). □

LEMMA 3.4. *Let B_n be a sequence of Borel subsets of ∂E. If $w_{B_n}(0) \ge \gamma > 0$ then there exist $\nu_n \in \mathcal{P}(B_n) \cap \mathcal{N}_1^E$ such that $h_{\nu_n}(0)$ and $G_E(h_{\nu_n}^\alpha)(0)$ are bounded. For every $1 < \gamma < \alpha$, $P_0 Z_{\nu_n}^\gamma$ are bounded and, consequently, Z_{ν_n} are uniformly P_0-integrable. The sequence Z_{ν_n} contains a subsequence convergent weakly in $L^1(P_0)$. Its limit Z has the properties: $P_0 Z > 0$ and $u_Z(x) = -\log P_x e^{-Z}$ is a moderate solution of the equation $\Delta u = u^\alpha$ in E. There exists a sequence $\hat{Z}_k$ which converges to Z P_x-a.s. for all $x \in E$. Moreover each $\hat{Z}_k$ is a convex combination of a finite numbers of Z_{ν_n}.*

PROOF. It follows from the bound **8**.(1.7) that

$$w_B(x) \le C(x)\,\mathrm{Cap}_x(B)^{1/(\alpha-1)} \tag{3.24}$$

where $C(x)$ does not depend on B. If $w_{B_n}(0) \ge \gamma$, then for all n, $\mathrm{Cap}_0(B_n) > \delta = [\gamma/C(0)]^{\alpha-1}$. By **2**.(4.1), there exists a compact $K_n \subset B_n$ such that $\mathrm{Cap}_x(K_n) > \delta/2$, and, by **6**.(1.3), $G_E(h_{\nu_n}^\alpha)(0) < 3/\delta$ for some $\nu_n \in \mathcal{P}(K_n)$. It follows from **2**.2.3.E that $\nu_n \in \mathcal{N}_1^E$.

We claim that there exists a constant c such that

$$h(0) \le c[G_E(h^\alpha)(0)]^{1/\alpha} \tag{3.25}$$

for every positive harmonic function h. Indeed, if the distance of 0 from ∂E is equal to 2ε, then, by the mean value property of harmonic functions,

$$h(0) = c_1^{-1}\int_{B_\varepsilon} h(y)\,dy \le (c_1c_2)^{-1}\int_{B_\varepsilon} g_E(0,y)h(y)\,dy \tag{3.26}$$

where $B_\varepsilon = \{x : |x| < \varepsilon\}$, c_1 is the volume of B_ε and $c_2 = \min g(0,y)$ over B_ε. By Hölder's inequality,

$$\int_{B_\varepsilon} g_E(0,y)h(y)\,dy \le [\int_{B_\varepsilon} g_E(0,y)h(y)^\alpha\,dy]^{1/\alpha}[\int_{B_\varepsilon} g_E(0,y)\,dy]^{1/\alpha'} \tag{3.27}$$

where $\alpha' = \alpha/(\alpha-1)$. Formula (3.25) follows from (3.26) and (3.27).

By (3.25),

$$h_{\nu_n}(0) \le c[G_E(h_{\nu_n}^\alpha)(0)]^{1/\alpha} \le c(3/\delta)^{1/\alpha}$$

and (3.17) implies that, for every $1 < \gamma < \alpha$, the sequence $P_0 Z_{\nu_n}^\gamma$ is bounded. This is sufficient for the uniform integrability of Z_{ν_n} (see, e. g., [Me66], p.19).

By the Dunford-Pettis criterion (see, e. g., [Me66], p. 20), Z_{ν_n} contains a subsequence that converges weakly in $L^1(P_0)$. By changing notation, we can assume that this subsequence coincide with Z_{ν_n}. The limit Z satisfies the condition $P_0 Z > 0$ because $P_0 Z_{\nu_n} \to P_0 Z$ and, by **3**.3.6.B,

$$P_0 Z_{\nu_n} = \int_{\partial E} k_E(0,y)\nu_n(dy) \geq \inf_{\partial E} k_E(0,y) > 0.$$

There exists a sequence $\tilde{Z}_m$ which converges to Z in $L^1(P_0)$ norm such that each $\tilde{Z}_m$ is a convex combination of a finite number of Z_{ν_n}. (See, e. g., [Ru73], Theorem 3.13.) A subsequence $\hat{Z}_k$ of $\tilde{Z}_m$ converges to Z P_0-a.s. By Theorem **5**.3.2, this implies that $\hat{Z}_k$ converges to Z P_x-a.s. for all $x \in E$. By **3**.3.4.B and **3**.3.4.C, u_Z is a moderate solution. □

3.3. Star domains belong to the class $\mathfrak{E}$. By Proposition 1.1, to prove Theorem 1.1 it is sufficient to demonstrate that every star domain E satisfies the condition 1.A.

We introduce a function

$$Q_r(y) = \hat{\Pi}_0^y \exp\{-\int_0^{\tau_E} u_r(\xi_t)^{\alpha-1}\, dt\}. \tag{3.28}$$

Consider, for every $\varepsilon > 0$ and every $0 < r < 1$, a partition of ∂E into two sets

$$A_{r,\varepsilon} = \{y \in \partial E : Q_r(y) \leq \varepsilon\} \quad \text{and } B_{r,\varepsilon} = \{y \in \partial E : Q_r(y) > \varepsilon\} \tag{3.29}$$

and denote by $I_{r,\varepsilon}$ and $J_{r,\varepsilon}$ the indicator functions of $A_{r,\varepsilon}$ and $B_{r,\varepsilon}$. Let us investigate the behavior, as $r \uparrow 1$, of functions

$$f_{r,\varepsilon} = V_E(u_r I_{r,\varepsilon}) \quad \text{and } g_{r,\varepsilon} = V_E(u_r J_{r,\varepsilon}). \tag{3.30}$$

We assume, as in 1.A, that

$$\mathrm{Tr}(u) = (\Lambda, \nu), \Lambda \subset \Gamma \subset \partial E \quad \text{and } \nu \text{ is concentrated on } \Gamma \tag{3.31}$$

and we prove:

LEMMA 3.5. *Put*

$$s_\varepsilon(x) = \limsup_{r\uparrow 1} g_{r,\varepsilon}(x).$$

For every $\varepsilon > 0$,

$$s_\varepsilon \leq w_\Gamma. \tag{3.32}$$

LEMMA 3.6. *Fix a relatively open subset O of ∂E which contains Γ and put*

$$C_{r,\varepsilon} = A_{r,\varepsilon} \cap (\partial E \setminus O), \quad q(\varepsilon) = \liminf_{r\uparrow 1} w_{C_{r,\varepsilon}}(0).$$

We have

$$\lim_{\varepsilon\downarrow 0} q(\varepsilon) = 0. \tag{3.33}$$

The property 1.A easily follows from these two lemmas. Indeed, $f_{r,\varepsilon}$ and $g_{r,\varepsilon}$ belong to $\mathcal{U}(E)$ by **2**.2.1.E. By **3**.3.5.C, $w_{A_{r,\varepsilon}} \leq w_O + w_{C_{r,\varepsilon}}$ because $A_{r,\varepsilon} \subset O \cup C_{r,\varepsilon}$. It follows from Lemma 3.6 that

$$\liminf_{\varepsilon \to 0} \liminf_{r \uparrow 1} w_{A_{r,\varepsilon}} \leq w_O(x).$$

Since this is true for all $O \supset \Gamma$,

$$\liminf_{\varepsilon \to 0} \liminf_{r \uparrow 1} w_{A_{r,\varepsilon}} \leq w_\Gamma(x) \tag{3.34}$$

by **3**.3.5.B.

Since $u \in \mathcal{U}(E)$ and $E_r \Subset E$, we have $V_{E_r}(u) = u$ in E_r and, by (3.2) and (3.3),

$$V_E(u_r) = u_r \quad \text{in } E_r. \tag{3.35}$$

By (3.35) and **2**.2.1.D,

$$u_r = V_E(u_r) \leq f_{r,\varepsilon} + g_{r,\varepsilon} \quad \text{in } E_r. \tag{3.36}$$

Since $\langle u_r 1_{A_{r,\varepsilon}}, X_E \rangle = 0$ on $\{X_E(A_{r,\varepsilon}) = 0\}$, we have

$$f_{r,\varepsilon} \leq -\log P_x\{X_E(A_{r,\varepsilon}) = 0\}$$

and, since X_E is supported, P_x-a.s., by $\mathcal{R}_E$, we get

$$f_{r,\varepsilon} \leq -\log P_x\{\mathcal{R}_E \cap A_{r,\varepsilon} = \emptyset\} = w_{A_{r,\varepsilon}}. \tag{3.37}$$

We conclude from (3.36), (3.32), (3.34) and (3.37) that

$$u(x) \leq \liminf_{\varepsilon \to 0} \liminf_{r \uparrow 1} w_{A_{r,\varepsilon}} + w_\Gamma(x) \leq 2 w_\Gamma(x). \tag{3.38}$$

By **3**.3.5.A, $Z_\Gamma = \mathrm{SBV}(w_\Gamma)$ takes only values 0 and ∞, and we have $Z_u \leq 2Z_\Gamma = Z_\Gamma$. which implies that $u \leq w_\Gamma$.

It remains to prove Lemma 3.5 and Lemma 3.6.

3.4. Proof of Lemma 3.5. Let us consider harmonic functions $h_{r,\varepsilon} = K_E(u_r J_{r,\varepsilon})$. By Jensen's inequality, $P_x e^{-\langle F, X_E \rangle} \geq e^{-P_x \langle F, X_E \rangle}$ for every $F \geq 0$. By applying this to $F = u_r J_{r,\varepsilon}$, we get

$$g_{r,\varepsilon} \leq h_{r,\varepsilon}. \tag{3.39}$$

First, we prove that

$$h_{r,\varepsilon}(0) \leq u(0)/\varepsilon. \tag{3.40}$$

By applying **3**.1.1.B to $v = u_r$ and $a(u) = u^{\alpha-1}$ we get

$$u_r(y) = \Pi_y u_r(\xi_{\tau_E}) Y \tag{3.41}$$

where

$$Y = \exp\left[-\int_0^{\tau_E} u_r(\xi_s)^{\alpha-1} ds\right].$$

By (3.41) and Lemma **3**.1.1,

$$u_r(0) = \Pi_0 u_r(\xi_{\tau_E}) \hat{\Pi}_0^{\xi_{\tau_E}} Y = K_E(u_r Q_r)(0).$$

Since $\varepsilon J_{r,\varepsilon} \le Q_r$, we have

$$\varepsilon h_{r,\varepsilon}(0) = K_E(\varepsilon u_r J_{r,\varepsilon})(0) \le K_E(u_r Q_r)(0) = u_r(0)$$

and (3.40) follows because $u_r(0) = r^\beta u(0) \le u(0)$.

To prove that (3.32) holds at $x \in E$, we choose a sequence $r_n \uparrow 1$ such that

$$g_{r_n,\varepsilon}(x) \to s_\varepsilon(x). \tag{3.42}$$

The bound (3.40) and well known properties of harmonic functions (see, e. g., **[D]**, 6.1.5.B and 6.1.5.C) imply that a subsequence of $h_{r_n,\varepsilon}$ tends to an element h_ε of $\mathcal{H}(E)$. By Lemma 3.1, this subsequence can be chosen in such a way that $g_{r_n,\varepsilon} \to g_\varepsilon \in \mathcal{U}(E)$. The bounds $g_{r,\varepsilon} \le h_{r,\varepsilon}$ imply that $g_\varepsilon \le h_\varepsilon$. Hence g_ε is a moderate solution and it is equal to u_μ for some $\mu \in \mathcal{N}_1^E$. By the definition of the fine trace, $\nu(B) \ge \mu'(B)$ for all $\mu' \in \mathcal{N}_1^E$ such that $\mu'(\Lambda) = 0$ and $u_{\mu'} \le u$. The restriction μ' of μ to $O = \partial E \setminus \Gamma$ satisfies these conditions. Indeed, $\mu' \in \mathcal{N}_1^E$ by **2**.2.3.A; $\mu'(\Lambda) = 0$ because $\Lambda \subset \Gamma$; finally, $u_{\mu'} \le u_\mu = g_\varepsilon \le u$ because $g_{r,\varepsilon} \le V_E(u_r) = u_r$ by **2**.2.1.B and (3.35). We conclude that $\mu'(O) \le \nu(O)$ and $\mu' = 0$ since $\nu(O) = 0$. Hence μ is supported by Γ and, by **2**.2.5.B, $g_\varepsilon(x) = u_\mu(x) \le w_\Gamma(x)$. By (3.42), $s_\varepsilon(x) = g_\varepsilon(x)$ which implies (3.32). □

3.5. Proof of Lemma 3.6. 1°. Clearly, $q(\varepsilon) \le q(\tilde{\varepsilon})$ for $\varepsilon < \tilde{\varepsilon}$. We need to show that $q(0+) = 0$. Suppose that this is not true and put $\gamma = q(0+)/2$. Consider a sequence $\varepsilon_n \downarrow 0$. Since $q(\varepsilon_n) \ge 2\gamma$, there exists $r_n > 1 - 1/n$ such that $w_{C_{r_n,\varepsilon_n}}(0) \ge \gamma$. We apply Lemma 3.4 to the sets $B_n = C_{r_n,\varepsilon_n}$. A sequence Z_{ν_n} defined in this lemma contains a weakly convergent subsequence. We redefine r_n and ε_n to make this subsequence identical with the sequence Z_{ν_n}.

2°. The next step is to prove that, if $Z_{\nu_n} \to Z$ weakly in $L^1(P_0)$, then the condition (3.31) implies

$$P_x Z e^{-Z_u} = 0 \tag{3.43}$$

for all $x \in E$. By Theorem **5**.3.2, since Z and Z_u are $\mathcal{F}_{\subset E-}$-measurable, it is sufficient to prove (3.43) for $x = 0$.

We apply Theorem **7**.4.1 to ν_n and $u_n = u_{r_n}$. By **7**.(4.1),

$$\begin{aligned} P_0 Z_{\nu_n} e^{-Z_{u_n}} &= e^{-u_n(0)} \Pi_0^{\nu_n} e^{-\Phi(u_n)} \\ &\le \Pi_0^{\nu_n} e^{-\Phi(u_n)} = \int_{\partial E} k_E(0,y) \hat{\Pi}_0^y e^{-\Phi(u_n)} \nu_n(dy) \end{aligned} \tag{3.44}$$

where Φ is defined by **7**.(4.3). Since $\psi'(u) = \alpha u^{\alpha-1} \ge u^{\alpha-1}$, we have

$$\hat{\Pi}_0^y e^{-\Phi(u_n)} \le Q_{r_n}(y)$$

where Q_r is defined by (3.28). Since $\nu_n \in \mathcal{P}(B_n)$ and since $Q_{r_n} \le \varepsilon_n$ on B_n, the right side in (3.44) does not exceed

$$\varepsilon_n \int_{\partial E} k_E(0,y) \nu_n(dy) = \varepsilon_n h_{\nu_n}(0).$$

By Lemma 3.4, the sequence $h_{\nu_n}(0)$ is bounded and therefore

$$P_0 Z_{\nu_n} e^{-Z_{u_n}} \to 0 \quad \text{as } n \to \infty. \tag{3.45}$$

Let $1 < \gamma < \alpha$. By Hölder's inequality,

$$|P_0 Z_{\nu_n}(e^{-Z_{u_n}} - e^{-Z_u})| \le (P_0 Z_{\nu_n}^{\gamma})^{1/\gamma} [P_0 |e^{-Z_{u_n}} - e^{-Z_u}|^{\gamma'}]^{1/\gamma'}$$

where $\gamma' = \gamma/(\gamma - 1) > 1$. By Lemma 3.4, the first factor is bounded. By Lemma 3.2, the second factor tends to 0. Hence

$$P_0 Z_{\nu_n} e^{-Z_{u_n}} - P_0 Z_{\nu_n} e^{-Z_u} \to 0. \tag{3.46}$$

Since $Z_{\nu_n} \to Z$ weakly in $L^1(P_0)$,

$$P_0 Z_{\nu_n} e^{-Z_u} \to P_0 Z e^{-Z_u}. \tag{3.47}$$

(3.43) follows from (3.45), (3.46) and (3.47).

3°. We deduce from (3.43) that

$$P_x\{Z = 0\} = 1 \tag{3.48}$$

which contradicts the relation $P_x Z > 0$ which is the part of Lemma 3.4. The contradiction proves that Lemma 3.6 is true.

Let Λ, Γ, ν be defined by (3.31) and let O be the set introduced in Lemma 3.6. We have

$$\Lambda \subset \Gamma \subset O. \tag{3.49}$$

By Lemma 3.4, $u_Z(x) = -\log P_x e^{-Z}$ is a moderate solution and therefore $u_Z = u_\mu$ for some $\mu \in \mathcal{N}_1^E$. The statement (3.48) will be proved if we show that $\mu = 0$.

It follows from (3.43) that $Z = 0$ P_x-a.s. on $\{Z_u < \infty\}$. Therefore $P_x\{Z_\mu \le Z_u\} = 1$ and

$$u_\mu \le u. \tag{3.50}$$

Note that ν_n is supported by $B_n \subset K = \partial E \setminus O$. By **2**.2.5.B, $u_{\nu_n} = 0$ on O and, by **1**.(1.5), $u_{\nu_n} \le w_K$. Therefore

$$Z_{\nu_n} = \mathrm{SBV}(u_{\nu_n}) \le \mathrm{SBV}(w_K) = Z_K.$$

By Lemma 3.4, there exists a sequence of $\hat{Z}_k$ such that $\hat{Z}_k \to Z$ P_x-a.s. for all $x \in E$ and each $\hat{Z}_k$ is a convex combination of a finite number of Z_{ν_n}. Therefore, P_x-a.s., $Z_\mu = Z \le Z_K$ and $u_\mu \le w_K$. By **2**.2.5.A, $w_K = 0$ on O. Hence $u_\mu = 0$ on O and, by **2**.2.3.D, $\mu(O) = 0$. By (3.49)

$$\mu(\Lambda) = 0. \tag{3.51}$$

By the definition of the trace (see **1**.(1.7)), (3.51) and (3.50) imply that $\mu \le \nu$. By the condition (3.31), $\nu(\partial E \setminus \Gamma) = 0$. Thus $\mu(\partial E \setminus \Gamma) = 0$ and $\mu(\partial E) \le \mu(O) + \mu(\partial E \setminus \Gamma) = 0$. $\square$

4. Notes

The material presented in this chapter was published first in [Dy04d]. The contents is close to the contents of Chapter 4 in [Ms04]. The most essential change needed to cover the case $\alpha \neq 2$ can be seen in our Lemmas 3.2, 3.3 and 3.4.

APPENDIX A

An elementary property of the Brownian motion

J.-F. Le Gall

We consider the Brownian motion (ξ_t, Π_x) in $\mathbb{R}^d$ and we give an upper bound for the maximal deviation of the path from the starting point x before the exit from a bounded domain E of class C^2.

LEMMA 0.1. *For every $x \in E$,*

$$\Pi_x\{\sup_{t \le \tau_E} |\xi_t - x| \ge r\} \le C\rho/r \tag{0.1}$$

where $\rho = d(x, \partial E)$.

PROOF. 1°. Clearly, (0.1) holds (with $C = 8$) for $r \le 8\rho$ and for $r \ge \operatorname{diam}(E)$. Therefore we can assume that $8\rho < r < \operatorname{diam}(E)$. Without any loss of generality we can assume that $\operatorname{diam}(E) = 2$.

2°. There exists a constant $a > 0$ such that every point $z \in \partial E$ can be touched from outside by a ball B of radius a. We consider a z such that $|x - z| = \rho$ and we place the origin at the center of B. Note that $|x| = a + \rho$. Put

$$\sigma_a = \inf\{t : |\xi_t| \le a\}, \qquad \tau^r = \inf\{t : |\xi_t - x| \ge r\}.$$

We have

$$\{\sup_{t \le \tau_E} |\xi_t - x| \ge r\} \subset \{\tau^r \le \tau_E\} \subset \{\tau^r \le \sigma_a\} \quad \Pi_x\text{-a.s.}$$

and therefore we get (0.1) if we prove that

$$\Pi_x\{\tau^r < \sigma_a\} \le C\rho/r. \tag{0.2}$$

3°. Let $\delta > 0$ be such that $16\delta(2 + a)^2 < 1$ (note that δ depends only on a). Let Γ be the cone

$$\Gamma = \{y \in \mathbb{R}^d : x \cdot y \ge (1 - \delta r^2)|x||y|\},$$

where $x \cdot y$ stands for the usual scalar product. Introduce the stopping time

$$U = \inf\{t \ge 0 : |\xi_t| > a + \frac{r}{2}\},$$
$$V = \inf\{t \ge 0 : \xi_t \notin \Gamma\}.$$

We first claim that

$$\{\tau^r < \sigma_a\} \subset \{U \wedge V < \sigma_a\}. \tag{0.3}$$

To prove (0.3), it is enough to verify that

$$\Gamma \cap \big(B(0, a + \frac{r}{2}) \backslash B(0,a)\big) \subset B(x,r)$$

($B(y,r) = B_r(y)$ is the open ball with radius r centered at y). However, if y belongs to the set $\Gamma \cap \big(B(0, a+\frac{r}{2})\backslash B(0,a)\big)$, then

$$\begin{aligned}|x-y|^2 &= |x|^2 + |y|^2 - 2x\cdot y \le |x|^2 + |y|^2 - 2(1-\delta r^2)|x||y| \\ &= (|y| - |x|)^2 + 2\delta r^2 |x||y| \le \frac{r^2}{4} + 2\delta r^2 (a+r)^2 \le r^2\end{aligned}$$

from our choice of δ. This gives our claim (0.3).

The lemma will then follow from (0.3) if we can get suitable bounds on both $\Pi_x\{U < \sigma_a\}$ and $\Pi_x\{V < \sigma_a\}$. First, from the formula for the scale function of the radial part of Brownian motion in $\mathbb{R}^d$,

$$\Pi_x\{U < \sigma_a\} = \frac{a^{2-d} - (a+\rho)^{2-d}}{a^{2-d} - (a+\frac{r}{2})^{2-d}} \le C' \, \frac{\rho}{r},$$

with a constant C' depending only on a.

To bound $\Pi_x\{V < \sigma_a\}$, consider the spherical domain $\Omega = \Gamma \cap S^d$ (where S^d is as usual the unit sphere in $\mathbb{R}^d$). Denote by λ the first eigenvalue of the operator $-\frac{1}{2}\Delta_{\mathrm{sph}}$ in Ω with Dirichlet boundary conditions (here Δ_{sph} is the spherical Laplacian), and let ϕ be the corresponding eigenfunction, which is strictly positive on Ω. Note that

$$\lambda \le \frac{c}{\delta r^2} \tag{0.4}$$

with a constant c depending only on the dimension d, and that ϕ attains its maximum at $x/|x|$ (by symmetry reasons).

Let $\nu = \frac{d}{2} - 1$. From the expression of the Laplacian in polar coordinates, it is immediately seen that the function

$$u(y) = |y|^{-\nu - \sqrt{\nu^2 + 2\lambda}} \, \phi(\frac{y}{|y|})$$

is harmonic in Γ. Since u vanishes on $\partial\Gamma$, the optional stopping theorem for the martingale $u(\xi_t)$ (at the stopping time $\sigma_a \wedge V$) implies

$$\begin{aligned}&|x|^{-\nu-\sqrt{\nu^2+\lambda}} \, \phi(\frac{x}{|x|}) = u(x) \\ &\qquad = \Pi_x\{u(\xi_{\sigma_a}) \, 1_{\{\sigma_a < V\}}\} \le \Pi_x\{\sigma_a < V\} \, a^{-\nu-\sqrt{\nu^2+2\lambda}} \sup_{z\in\Omega} \phi(z).\end{aligned}$$

Recalling that ϕ attains its maximum at $x/|x|$, we obtain

$$\Pi_x\{\sigma_a < V\} \ge \Big(\frac{a}{a+\rho}\Big)^{\nu + \sqrt{\nu^2+2\lambda}},$$

and thus

$$\Pi_x\{V < \sigma_a\} \le 1 - \Big(\frac{a}{a+\rho}\Big)^{\nu+\sqrt{\nu^2+2\lambda}}.$$

From this inequality and the bound (0.4), we easily derive the existence of a constant C'' depending only on a such that

$$\Pi_x\{V < \sigma_a\} \leq C''(\frac{\rho}{r}).$$

This completes the proof of the lemma. □

APPENDIX B

Relations between Poisson and Bessel capacities

I. E. Verbitsky

We show that the Poisson capacities $\mathrm{Cap}(\Gamma)$ introduced in Chapter **6** are equivalent to $[\mathrm{Cap}_{l,p}(\Gamma)]^{\frac{1}{p-1}}$, where $l = \frac{2}{\alpha}$, $p = \alpha'$ and $\mathrm{Cap}_{l,p}$ are the Bessel capacities (used in [MV03], [MV04]). It is easy to show that, if $1 < d < \frac{\alpha+1}{\alpha-1}$, then, for every nonempty set Γ on the boundary of a bounded smooth domain, both $\mathrm{Cap}(\Gamma)$ and $\mathrm{Cap}_{l,p}(\Gamma)$ are bounded from above and from below by strictly positive constants. Therefore it suffices to consider only the supercritical case $d \geq \frac{\alpha+1}{\alpha-1}$.

By using the straightening of the boundary described in the Introduction, one can reduce the case of the Poisson capacity Cap on the boundary $\tilde{E}$ of a bounded $C^{2,\lambda}$-domain E in $\mathbb{R}^d$ to the capacity $\widetilde{\mathrm{Cap}}$ on the boundary $E_0 = \{x = (x_1, \dots, x_d) : x_d = 0\}$ of the half-space $E_+ = \mathbb{R}^{d-1} \times (0, \infty)$ (see Section **6**.3). We will use the notation **6**.(3.1)-**6**.(3.2):

$$\begin{aligned}
&\mathbb{E} = \{x = (x_1, \dots, x_d) : 0 < x_d < 1\},\\
&r(x) = d(x, E_0) = x_d,\\
&\tilde{k}(x, y) = r(x)|x-y|^{-d}, \quad x \in \mathbb{E},\, y \in E_0,\\
&\tilde{m}(dx) = r(x)dx, \quad x \in \mathbb{E}.
\end{aligned}$$

For $\nu \in \mathcal{M}(E_0)$, we set

$$(\widetilde{K}\nu)(x) = \int_{E_0} \tilde{k}(x, y)\nu(dy), \qquad \widetilde{\mathcal{E}}(\nu) = \int_{\mathbb{E}} (\widetilde{K}\nu)^\alpha \, d\tilde{m}. \tag{0.5}$$

The capacity $\widetilde{\mathrm{Cap}}$ on E_0 associated with $(\tilde{k}, \tilde{m})$ is given by any one of the equivalent definitions **6**.(1.3), **6**.(1.4), **6**.(1.5). According to the second definition (which will be the most useful for us),

$$\widetilde{\mathrm{Cap}}(\Gamma) = [\sup\{\nu(\Gamma) : \nu \in \mathcal{M}(\Gamma), \quad \widetilde{\mathcal{E}}(\nu) \leq 1\}]^\alpha. \tag{0.6}$$

The Bessel capacity on E_0 can be defined in terms of the Bessel kernel [1]

$$G_l(x) = \frac{1}{(4\pi)^{l/2}\Gamma(l/2)} \int_0^\infty t^{\frac{l-d+1}{2}} e^{-\frac{\pi|x|^2}{t} - \frac{t}{4\pi}} \frac{dt}{t}, \quad x \in E_0.$$

[1] See, for instance, [AH96] or [Ma85].

For every $l > 0, p > 1$ and $\Gamma \subset E_0$,

$$\operatorname{Cap}_{l,p}(\Gamma) = \inf\{\int_{E_0} [f(x)]^p \, dx : \; f \in \mathcal{B}(E_0), \quad G_l f \geq 1 \quad \text{on } \Gamma\} \tag{0.7}$$

where

$$G_l f(x) = G_l \star f(x) = \int_{E_0} G_l(x-t)\, f(t)\, dt.$$

We need the asymptotics ([AH96], Section 1.2.5) [2]

$$G_l(x) \asymp |x|^{l-d+1}, \quad \text{as } |x| \to 0, \quad 0 < l < d-1, \tag{0.8}$$

$$G_l(x) \asymp \log \frac{1}{|x|}, \quad \text{as } |x| \to 0, \quad l = d-1, \tag{0.9}$$

$$G_l(x) \asymp 1, \quad \text{as } |x| \to 0, \quad l > d-1, \tag{0.10}$$

$$G_l(x) \asymp |x|^{(l-d)/2} e^{-|x|}, \quad \text{as } |x| \to \infty, \quad l > 0. \tag{0.11}$$

THEOREM 0.1. *Suppose that $\alpha > 1$ and $d \geq \frac{\alpha+1}{\alpha-1}$. Then there exist strictly positive constants C_1 and C_2 such that, for all $\Gamma \subset E_0$,*

$$C_1 \, [\operatorname{Cap}_{\frac{2}{\alpha},\alpha'}(\Gamma)]^{\alpha-1} \leq \widetilde{\operatorname{Cap}}(\Gamma) \leq C_2 \, [\operatorname{Cap}_{\frac{2}{\alpha},\alpha'}(\Gamma)]^{\alpha-1}. \tag{0.12}$$

To prove Theorem 0.1, we need a dual definition of the Bessel capacity $\operatorname{Cap}_{l,p}$. For $\nu \in \mathcal{M}(E_0)$, the (l,p)-energy $\mathcal{E}_{l,p}(\nu)$ is defined by

$$\mathcal{E}_{l,p}(\nu) = \int_{E_0} (G_l \nu)^{p'} \, dx. \tag{0.13}$$

Then $\operatorname{Cap}_{l,p}(\Gamma)$ can be defined equivalently ([AH96, Section 2.2], [Ma85]) by

$$\operatorname{Cap}_{l,p}(\Gamma) = [\sup\{\nu(\Gamma) : \; \nu \in \mathcal{M}(\Gamma), \quad \mathcal{E}_{l,p}(\nu) \leq 1\}]^p. \tag{0.14}$$

For $l > 0$, $p > 1$, define the (l,p)-Poisson energy of $\nu \in \mathcal{M}(E_0)$ by

$$\widetilde{\mathcal{E}}_{l,p}(\nu) = \int_{\mathbb{E}} [\widetilde{K}\nu(x)]^{p'} \, r(x)^{lp'-1} \, dx. \tag{0.15}$$

LEMMA 0.2. *Let $p > 1$ and $0 < l < d-1$. Then there exist strictly positive constants C_1 and C_2 which depend only on l, p, and d such that, for all $\nu \in \mathcal{M}(E_0)$,*

$$C_1 \, \mathcal{E}_{l,p}(\nu) \leq \widetilde{\mathcal{E}}_{l,p}(\nu) \leq C_2 \, \mathcal{E}_{l,p}(\nu). \tag{0.16}$$

PROOF. We first prove the upper estimate in (0.16).

PROPOSITION 0.1. *Let $\alpha \geq 1$. Suppose $\phi : \; (0,\infty) \to (0,\infty)$ is a measurable function such that*

$$\phi(y) \leq c \int_0^y \phi(s) \frac{ds}{s}, \quad y > 0. \tag{0.17}$$

[2] We write $F(x) \asymp G(x)$ as $x \to a$ if $\frac{F(x)}{G(x)} \to c$ as $x \to a$ where c is a strictly positive constant.

Then,

$$\int_0^y [\phi(s)]^\alpha \frac{ds}{s} \le c^{\alpha-1} \left(\int_0^y \phi(s) \frac{ds}{s} \right)^\alpha, \quad y > 0. \tag{0.18}$$

PROOF. We estimate:

$$\int_0^y \phi(s) \left[\frac{\phi(s)}{\int_0^s \phi(t) \frac{dt}{t}} \right]^{\alpha-1} \frac{ds}{s} \le c^{\alpha-1} \int_0^y \phi(s) \frac{ds}{s}.$$

Since $s < y$ in the preceding inequality, one can put $\int_0^y \phi(t) \frac{dt}{t}$ in place of $\int_0^s \phi(t) \frac{dt}{t}$ on the left-hand side, which gives (0.18). □

Let $x = (x', y)$ where $x' = (x_1, \ldots, x_{d-1})$, and $y = x_d = r(x)$. We now set $\phi(y) = y^l \widetilde{K}\nu(x', y)$. It follows from (0.5) and the expression for $\tilde{k}(x, y)$ that, if $\frac{y}{2} \le s \le y$, then

$$\phi(y) \le c\, \phi(s)$$

where c depends only on d. Hence, ϕ satisfies (0.17). Applying Proposition 0.1 with $\alpha = p'$, we have

$$\int_0^1 [\widetilde{K}\nu(x', y)]^{p'} y^{lp'} \frac{dy}{y} \le c^{p'-1} \left(\int_0^1 \widetilde{K}\nu(x', y)\, y^l \frac{dy}{y} \right)^{p'}.$$

Integrating both sides of the preceding inequality over E_0, we obtain

$$\begin{aligned} \widetilde{\mathcal{E}}_{l,p}(\nu) &= \int_{\mathbb{E}} [\widetilde{K}\nu(x)]^{p'} r(x)^{lp'-1} dx \\ &\le c^{p'-1} \int_{E_0} \left(\int_0^1 \widetilde{K}\nu(x', y)\, y^l \frac{dy}{y} \right)^{p'} dx'. \end{aligned}$$

By Fubini's theorem,

$$\int_0^1 \widetilde{K}\nu(x', y)\, y^l \frac{dy}{y} = \int_{E_0} \int_0^1 \frac{y^l}{[(x'-t)^2 + y^2]^{\frac{d}{2}}}\, dy\, \nu(dt).$$

For $|x' - t| \ge 1$, we will use the estimate

$$\int_0^1 \frac{y^l}{[(x'-t)^2 + y^2]^{\frac{d}{2}}} dy \le \frac{C}{|x'-t|^d}. \tag{0.19}$$

For $|x' - t| < 1$,

$$\int_0^1 \frac{y^l}{[(x'-t)^2 + y^2]^{\frac{d}{2}}} dy \le \frac{C}{|x'-t|^{d-l-1}},$$

in the case $0 < l < d - 1$; the left-hand side of the preceding inequality is bounded by $C \log \frac{2}{|x'-t|}$ if $l = d - 1$, and by C if $l > d - 1$, where C depends only on l and d. Using asymptotics (0.8)–(0.10), we rewrite the preceding estimates in the form

$$\int_0^1 \frac{y^l}{[(x'-t)^2 + y^2]^{\frac{d}{2}}} dy \le C\, G_l(|x'-t|), \quad |x'-t| < 1. \tag{0.20}$$

Thus, by (0.19) and (0.20),

$$\widetilde{\mathcal{E}}_{l,p}(\nu) \le C \int_{E_0} \left(\int_{|x'-t|<1} G_l(|x'-t|)\, \nu(dt) \right)^{p'} dx' + C \int_{E_0} \left(\int_{|x'-t|\ge 1} \frac{\nu(dt)}{|x'-t|^d} \right)^{p'} dx'.$$

The first term on the right is obviously bounded by $\mathcal{E}_{l,p}(\nu)$. To estimate the second term, we notice that

$$\int_{|x'-t|\ge 1} \frac{\nu(dt)}{|x'-t|^d} \le C \int_1^\infty \frac{\nu(B(x',r))}{r^d} \frac{dr}{r} \le C \sup_{r\ge 1} \frac{\nu(B(x',r))}{r^{d-1}}. \tag{0.21}$$

We will need the Hardy–Littlewod maximal function on $E_0 = \mathbb{R}^{d-1}$:

$$M(f)(x) = \sup_{r>0} \frac{1}{r^{d-1}} \int_{B(x,r)} |f(t)|\, dt, \quad x \in E_0,$$

which is a bounded operator on $L^p(E_0)$ for $1 < p < \infty$. [3]

Hence,

$$||M(G_l\nu)||_{L^{p'}(E_0)} \le C\, ||G_l\nu||_{L^{p'}(E_0)} = C\, \mathcal{E}_{l,p}(\nu)^{\frac{1}{p'}},$$

where C depends only on p. It is easy to see that

$$M(G_l\nu)(x') \ge C \sup_{r\ge 1} \frac{\nu(B(x',r))}{r^{d-1}}, \quad x' \in E_0.$$

Thus, by the preceding estimates and (0.21), it follows

$$\int_{E_0} \left(\int_{|x'-t|\ge 1} \frac{\nu(dt)}{|x'-t|^d} \right)^{p'} dx' \le C\, ||M(G_l\nu)||^{p'}_{L^{p'}(E_0)} \le C\, \mathcal{E}_{l,p}(\nu),$$

where C depends only on l, p, and d. This completes the proof of the upper estimate in (0.16).

To prove the lower estimate, notice that, for every $0 < r < 1$,

$$\int_0^1 [\widetilde{K}\nu(x',y)]^{p'} y^{lp'} \frac{dy}{y} \ge \int_{\frac{r}{2}}^1 \left[\int_{|x'-t|<r} \frac{y^{l+1}\, \nu(dt)}{(|x'-t|^2 + y^2)^{\frac{d}{2}}} \right]^{p'} \frac{dy}{y}$$

$$\ge C\, [\nu(B(x',r))]^{p'} \int_{\frac{r}{2}}^1 y^{(l+1-d)p'} \frac{dy}{y} \ge C\, [r^{l+1-d}\, \nu(B(x',r))]^{p'},$$

provided $0 < l < d-1$. This implies

$$\int_0^1 [\widetilde{K}\nu(x',y)]^{p'} y^{lp'} \frac{dy}{y} \ge C\, M_l(\nu)(x')^{p'}, \quad x' \in E_0,$$

where

$$M_l(\nu)(x') = \sup_{0<r<1} r^{l-d+1}\, \nu(B(x',r)), \quad x' \in E_0.$$

[3] See, e. g., [AH96], Theorem 1.1.1.

Consequently,

$$\widetilde{\mathcal{E}}_{l,p}(\nu) \geq C\, ||M_l(\nu)||^{p'}_{L^{p'}(E_0)}. \tag{0.22}$$

By a theorem of Muckenhoupt and Wheeden [MW74] (or, more precisely, its inhomogeneous version [AH96], Theorem 3.6.2),

$$||M_l(\nu)||^{p'}_{L^{p'}(E_0)} \geq C\, ||G_l(\nu)||^{p'}_{L^{p'}(E_0)} = C\, \mathcal{E}_{l,p}(\nu). \tag{0.23}$$

Thus,

$$\widetilde{\mathcal{E}}_{l,p}(\nu) \geq C||M_l(\nu)||^{p'}_{L^{p'}(E_0)} \geq C\, \mathcal{E}_{l,p}(\nu),$$

which gives the lower estimate in (0.16). The proof of Lemma 0.2 is complete. □

We now complete the proof of Theorem 0.1. The condition $d \geq \frac{\alpha+1}{\alpha-1}$ implies that $0 < l \leq \frac{d-1}{p} < d-1$ for $l = \frac{2}{\alpha}$ and $p = \alpha'$. By Lemma 0.2,

$$C_1\, \mathcal{E}_{\frac{2}{\alpha},\alpha'}(\nu) \leq \widetilde{\mathcal{E}}(\nu) \leq C_2\, \mathcal{E}_{\frac{2}{\alpha},\alpha'}(\nu)$$

where $\widetilde{\mathcal{E}}(\nu)$ is defined by (0.5), and C_1 and C_2 are strictly positive constants which depend only on α and d. By combining the preceding inequality with definitions (0.6), (0.14), we complete the proof. □

Notes

Lemma 0.2 holds for all $l > 0$. The restriction $0 < l < d-1$ was used only in the proof of the lower estimate (0.16). The case $l \geq d-1$ can be treated in a slightly different way using, for instance, estimates of the energy in terms of nonlinear potentials from [COV04].

Lemma 0.2 may be also deduced from the following two facts. First, if $||\nu||_{B^{-l,p'}}$ denotes the norm of a distribution $\nu \in \mathcal{S}(E_0)$ in the (inhomogeneous) Besov space $B^{-l,p'} = [B^{l,p}]^*$ on E_0 ($l > 0$, $p > 1$), then

$$||\nu||^{p'}_{B^{-l,p'}} \asymp \widetilde{\mathcal{E}}_{l,p}(\nu) = \int_{\mathbb{E}} |\widetilde{K} \star \nu(x)|^{p'} r(x)^{lp'-1}\, dx,$$

where $\widetilde{K} \star \nu$ is a harmonic extension of ν to E_+. Such characterizations of $B^{l,p}$ spaces have been known to experts for a long time, but complete proofs in the case of negative l are not so easy to find in the literature. We refer to [BHQ79] where analogous results were obtained for homogeneous Besov spaces $\dot{B}^{l,p}$ ($l \in \mathbb{R}$, $p > 0$). In the proof above, we used instead direct estimates of $\widetilde{\mathcal{E}}_{l,p}(\nu)$ for nonnegative ν.

Secondly, for nonnegative ν,

$$||\nu||^{p'}_{B^{-l,p'}} \asymp ||\nu||^{p'}_{W^{-l,p'}} \asymp \mathcal{E}_{l,p}(\nu),$$

where $W^{-l,p'} = [W^{l,p}]^*$ is the dual Sobolev space on E_0. This fact, first observed by D. Adams, is a consequence of Wolff's inequality which appeared in [HW83]. (See [AH96], Sections 4.5 and 4.9 for a thorough discussion of these estimates, their history and applications).

Thus, an alternative proof of Lemma 0.2 can be based on Wolff's inequality, which in its turn may be deduced from the Muckhenhoupt–Wheeden fractional maximal theorem used above. We note that the original proof of Wolff's inequality given in [HW83] has been generalized to arbitrary radially decreasing kernels [COV04], and has applications to semilinear elliptic equations [KV99].

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